Convergent Filter Bases

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Summary. We are inspired by the work of Henri Cartan [10], Bourbaki [10] (TG. I Filtres) and Claude Wagschal [34]. We define the base of filter, image filter, convergent filter bases, limit filter and the filter base of tails (fr: filtre des sections).

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The notation and terminology used in this paper have been introduced in the following articles: [24], [1], [2], [33], [20], [18], [28], [11], [12], [13], [29], [3], [37], [25], [26], [4], [17], [30], [5], [14], [23], [35], [36], [22], [31], [6], [7], [19], [27], and [15].

1. Filters – Set-Theoretical Approach

From now on $X$ denotes a non empty set, $\mathcal{F}$ denotes a filter of $X$, and $S$ denotes a family of subsets of $X$.

Let $X$ be a set and $S$ be a family of subsets of $X$. We say that $S$ is upper if and only if

(Def. 1) for every subsets $Y_1, Y_2$ of $X$ such that $Y_1 \in S$ and $Y_1 \subseteq Y_2$ holds $Y_2 \in S$.

Let us note that there exists a $\cap$-closed family of subsets of $X$ which is non empty and there exists a non empty, $\cap$-closed family of subsets of $X$ which is upper.

Let $X$ be a non empty set. Let us note that there exists a non empty, upper, $\cap$-closed family of subsets of $X$ which has non empty elements.

Now we state the propositions:
(1) $S$ is a non empty, upper, $\cap$-closed family of subsets of $X$ with non empty elements if and only if $S$ is a filter of $X$.

(2) Let us consider non empty sets $X_1$, $X_2$, a filter $\mathcal{F}_1$ of $X_1$, and a filter $\mathcal{F}_2$ of $X_2$. Then the set of all $f_1 \times f_2$ where $f_1$ is an element of $\mathcal{F}_1$, $f_2$ is an element of $\mathcal{F}_2$ is a non empty family of subsets of $X_1 \times X_2$.

Let $X$ be a non empty set. We say that $X$ is $\cap$-finite closed if and only if

(Def. 2) for every finite, non empty subset $S_1$ of $X$, $\bigcap S_1 \in X$.

One can check that there exists a non empty set which is $\cap$-finite closed.

Now we state the proposition:

(3) Let us consider a non empty set $X$. If $X$ is $\cap$-finite closed, then $X$ is $\cap$-closed.

Note that every non empty set which is $\cap$-finite closed is also $\cap$-closed.

(4) Let us consider a set $X$, and a family $S$ of subsets of $X$. Then $S$ is $\cap$-closed and $X \in S$ if and only if \( \text{FinMeetCl}(S) \subseteq S \).

(5) Let us consider a non empty set $X$, and a non empty subset $A$ of $X$.

Then $\{B, \text{where } B \text{ is a subset of } X: A \subseteq B\}$ is a filter of $X$.

Let $X$ be a non empty set. Note that every filter of $X$ is $\cap$-closed.

(6) Let us consider a set $X$, and a family $B$ of subsets of $X$. If $B = \{X\}$, then $B$ is upper.

(7) Let us consider a non empty set $X$, and a filter $\mathcal{F}'$ of $X$. Then $\mathcal{F}' \neq 2^X$.

Let $X$ be a non empty set. The functor $\text{Filt}(X)$ yielding a non empty set is defined by the term

(Def. 3) the set of all $\mathcal{F}'$ where $\mathcal{F}'$ is a filter of $X$.

Let $I$ be a non empty set and $M$ be a $(\text{Filt}(X))$-valued many sorted set indexed by $I$. The intersection of the family of filters $M$ yielding a filter of $X$ is defined by the term

(Def. 4) $\bigcap \text{rng } M$.

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be filters of $X$. We say that $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$ if and only if

(Def. 5) $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

One can verify that the predicate is reflexive. We say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ if and only if

(Def. 6) $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

Observe that the predicate is reflexive.

Now we state the propositions:

(8) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a filter $\mathcal{F}$ of $X$.

Suppose $\mathcal{F} = \{X\}$. Then $\mathcal{F}$ is coarser than $\mathcal{F}'$. 
(9) Let us consider a non empty set $X$, a non empty set $I$, a $(\text{Filt}(X))$-valued many sorted set $M$ indexed by $I$, an element $i$ of $I$, and a filter $\mathcal{F}'$ of $X$. Suppose $\mathcal{F}' = M(i)$. Then the intersection of the family of filters $M$ is coarser than $\mathcal{F}'$.

(10) Let us consider a set $X$, and a family $S$ of subsets of $X$. Suppose $\text{FinMeetCl}(S)$ has non empty elements. Then $S$ has non empty elements.

(11) Let us consider a non empty set $X$, a family $G$ of subsets of $X$, and a filter $\mathcal{F}'$ of $X$. Suppose $G \subseteq \mathcal{F}'$. Then

(i) $\text{FinMeetCl}(G) \subseteq \mathcal{F}'$, and

(ii) $\text{FinMeetCl}(G)$ has non empty elements.

The theorem is a consequence of (4).

Let $X$ be a non empty set, $\mathcal{F}'$ be a filter of $X$, and $B$ be a non empty subset of $\mathcal{F}'$. We say that $B$ is filter basis if and only if

(Def. 7) for every element $f$ of $\mathcal{F}'$, there exists an element $b$ of $B$ such that $b \subseteq f$.

Now we state the proposition:

(12) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a non empty subset $B$ of $\mathcal{F}'$. Then $\mathcal{F}'$ is coarser than $B$ if and only if $B$ is filter basis.

Let $X$ be a non empty set and $\mathcal{F}'$ be a filter of $X$. Observe that there exists a non empty subset of $\mathcal{F}'$ which is filter basis.

A generalized basis of $\mathcal{F}'$ is a filter basis, non empty subset of $\mathcal{F}'$. Now we state the proposition:

(13) Let us consider a non empty set $X$. Then every filter of $X$ is a generalized basis of $\mathcal{F}'$.

Let $X$ be a set and $B$ be a family of subsets of $X$. The functor $[B]$ yielding a family of subsets of $X$ is defined by

(Def. 8) for every subset $x$ of $X$, $x \in [B]$ iff there exists an element $b$ of $B$ such that $b \subseteq x$.

Now we state the propositions:

(14) Let us consider a set $X$, and a family $S$ of subsets of $X$. Then $[S] = \{x, \text{where } x \text{ is a subset of } X : \text{there exists an element } b \text{ of } S \text{ such that } b \subseteq x\}$.

(15) Let us consider a set $X$, and an empty family $B$ of subsets of $X$. Then $[B] = 2^X$.

(16) Let us consider a set $X$, and a family $B$ of subsets of $X$. If $\emptyset \in B$, then $[B] = 2^X$. 

2. Filters – Lattice-Theoretical Approach

Now we state the propositions:

(17) Let us consider a set $X$, a non empty family $B$ of subsets of $X$, and a subset $L$ of $2^X$. If $B = L$, then $[B] = \uparrow L$.

(18) Let us consider a set $X$, and a family $B$ of subsets of $X$. Then $B \subseteq [B]$.

Let $X$ be a set and $B_1, B_2$ be families of subsets of $X$. We say that $B_1$ and $B_2$ are equivalent generators if and only if

\begin{itemize}
  \item (Def. 9) for every element $b_1$ of $B_1$, there exists an element $b_2$ of $B_2$ such that $b_2 \subseteq b_1$ and for every element $b_2$ of $B_2$, there exists an element $b_1$ of $B_1$ such that $b_1 \subseteq b_2$.
\end{itemize}

Let us note that the predicate is reflexive and symmetric.

Let us consider a set $X$ and families $B_1, B_2$ of subsets of $X$.

Let us assume that $B_1$ and $B_2$ are equivalent generators. Now we state the propositions:

(19) $[B_1] \subseteq [B_2]$.

(20) $[B_1] = [B_2]$.

Let $X$ be a non empty set, $F'$ be a filter of $X$, and $B$ be a non empty subset of $F'$. The functor $\# B$ yielding a non empty family of subsets of $X$ is defined by the term

\begin{itemize}
  \item (Def. 10) $B$.
\end{itemize}

Now we state the propositions:

(21) Let us consider a non empty set $X$, a filter $F'$ of $X$, and a generalized basis $B$ of $F'$. Then $F' = [\# B]$.

(22) Let us consider a non empty set $X$, a filter $F'$ of $X$, and a family $B$ of subsets of $X$. If $F' = [B]$, then $B$ is a generalized basis of $F'$.

(23) Let us consider a non empty set $X$, a filter $F'$ of $X$, a generalized basis $B$ of $F'$, a family $S$ of subsets of $X$, and a subset $S_1$ of $F'$. Suppose $S = S_1$ and $\# B$ and $S$ are equivalent generators. Then $S_1$ is a generalized basis of $F'$. The theorem is a consequence of (19), (21), and (22).

(24) Let us consider a non empty set $X$, a filter $F'$ of $X$, and generalized bases $B_1, B_2$ of $F'$. Then $\# B_1$ and $\# B_2$ are equivalent generators. The theorem is a consequence of (21).

Let $X$ be a set and $B$ be a family of subsets of $X$. We say that $B$ is quasi basis if and only if

\begin{itemize}
  \item (Def. 11) for every elements $b_1, b_2$ of $B$, there exists an element $b$ of $B$ such that $b \subseteq b_1 \cap b_2$.
\end{itemize}
Let \( X \) be a non empty set. Let us note that there exists a non empty family of subsets of \( X \) which is quasi basis and there exists a non empty, quasi basis family of subsets of \( X \) which has non empty elements.

A filter base of \( X \) is a non empty, quasi basis family of subsets of \( X \) with non empty elements. Now we state the proposition:

(25) Let us consider a non empty set \( X \), and a filter base \( B \) of \( X \). Then \([B]\) is a filter of \( X \).

Let \( X \) be a non empty set and \( B \) be a filter base of \( X \). The functor \([B]\) yielding a filter of \( X \) is defined by the term

(Def. 12) \([B]\).

Now we state the propositions:

(26) Let us consider a non empty set \( X \), and filter bases \( B_1, B_2 \) of \( X \). Suppose \([B_1] = [B_2]\). Then \( B_1 \) and \( B_2 \) are equivalent generators.

(27) Let us consider a non empty set \( X \), a filter base \( F \) of \( X \), and a filter \( F' \) of \( X \). Suppose \( F \subseteq F' \). Then \([F]\) is coarser than \([F']\).

(28) Let us consider a non empty set \( X \), and a family \( G \) of subsets of \( X \). Suppose \( \text{FinMeetCl}(G) \) has non empty elements. Then

(i) \( \text{FinMeetCl}(G) \) is a filter base of \( X \), and

(ii) there exists a filter \( F' \) of \( X \) such that \( \text{FinMeetCl}(G) \subseteq F' \).

The theorem is a consequence of (4).

(29) Let us consider a non empty set \( X \), and a filter \( F' \) of \( X \). Then every generalized basis of \( F' \) is a filter base of \( X \).

(30) Let us consider a non empty set \( X \). Then every filter base of \( X \) is a generalized basis of \([B]\).

(31) Let us consider a non empty set \( X \), a filter \( F' \) of \( X \), a generalized basis \( B \) of \( F' \), and a subset \( L \) of \( 2^X \). If \( L = \# B \), then \( F' = \uparrow L \). The theorem is a consequence of (21) and (17).

(32) Let us consider a non empty set \( X \), a filter base \( B \) of \( X \), and a subset \( L \) of \( 2^X \). If \( L = B \), then \([B] = \uparrow L \).

(33) Let us consider a non empty set \( X \), filters \( F_1, F_2 \) of \( X \), a generalized basis \( B_1 \) of \( F_1 \), and a generalized basis \( B_2 \) of \( F_2 \). Then \( F_1 \) is coarser than \( F_2 \) if and only if \( B_1 \) is coarser than \( B_2 \). The theorem is a consequence of (21).

(34) Let us consider non empty sets \( X, Y \), a function \( f \) from \( X \) into \( Y \), a filter \( F' \) of \( X \), and a generalized basis \( B \) of \( F' \). Then

(i) \( f^\circ (\# B) \) is a filter base of \( Y \), and

(ii) \([f^\circ (\# B)]\) is a filter of \( Y \), and
(iii) \([f^\circ(\# B)] = \{M, \text{where } M \text{ is a subset of } Y : f^{-1}(M) \in F'\}\).

**Proof:** Set \(F = f^\circ(\# B)\). \(F\) is a quasi basis, non empty family of subsets of \(Y\) by (29), \([35] (123), (121)]\). \(F\) has non empty elements by \([35] (118)]\).

\([F] = \{M, \text{where } M \text{ is a subset of } Y : f^{-1}(M) \in F'\}\) by \([35] (143)], [12] (42)), (21), [35] (123)]. \(\square\)

Let \(X, Y\) be non empty sets, \(f\) be a function from \(X\) into \(Y\), and \(F'\) be a filter of \(X\). The image of filter \(F'\) under \(f\) yielding a filter of \(Y\) is defined by the term

(Def. 13) \(\{M, \text{where } M \text{ is a subset of } Y : f^{-1}(M) \in F'\}\).

Now we state the propositions:

(35) Let us consider non empty sets \(X, Y\), a function \(f\) from \(X\) into \(Y\), and a filter \(F'\) of \(X\). Then

(i) \(f^\circ F'\) is a filter base of \(Y\), and

(ii) \([f^\circ F'] = \text{the image of filter } F' \text{ under } f\).

The theorem is a consequence of (13) and (34).

(36) Let us consider a non empty set \(X\), and a filter base \(B\) of \(X\). If \(B = [B]\), then \(B\) is a filter of \(X\).

(37) Let us consider non empty sets \(X, Y\), a function \(f\) from \(X\) into \(Y\), a filter \(F'\) of \(X\), and a generalized basis \(B\) of \(F'\). Then

(i) \(f^\circ(\# B)\) is a generalized basis of the image of filter \(F'\) under \(f\), and

(ii) \([f^\circ(\# B)] = \text{the image of filter } F' \text{ under } f\).

The theorem is a consequence of (34) and (30).

(38) Let us consider non empty sets \(X, Y\), a function \(f\) from \(X\) into \(Y\), and filter bases \(B_1, B_2\) of \(X\). Suppose \(B_1\) is coarser than \(B_2\). Then \([B_1]\) is coarser than \([B_2]\). The theorem is a consequence of (30) and (33).

(39) Let us consider non empty sets \(X, Y\), a function \(f\) from \(X\) into \(Y\), and a filter \(F'\) of \(X\). Then \(f^\circ F'\) is a filter of \(Y\) if and only if \(Y = \text{rng } f\).

**Proof:** Reconsider \(f_3 = f^\circ F'\) as a filter base of \(Y\). \([f_3] \subseteq f_3\) by \([35] (143)], [11] (76), (77)]). \(\square\)

(40) Let us consider a non empty set \(X\), a non empty subset \(A\) of \(X\), a filter \(F'\) of \(A\), and a generalized basis \(B\) of \(F'\). Then

(i) \((\triangle A)^\circ(\# B)\) is a filter base of \(X\), and

(ii) \([(\triangle A)^\circ(\# B)] = \text{a filter of } X, \text{and}

(iii) \([(\triangle A)^\circ(\# B)] = \{M, \text{where } M \text{ is a subset of } X : (\triangle A)^{-1}(M) \in F'\}\).

Let \(L\) be a non empty relational structure. The functor \(\text{Tails}(L)\) yielding a non empty family of subsets of \(L\) is defined by the term
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(Def. 14) the set of all $\uparrow i$ where $i$ is an element of $L$.

Now we state the proposition:

(41) Let us consider a non empty, transitive, reflexive relational structure $L$. Suppose $\Omega_L$ is directed. Then $[\text{Tails}(L)]$ is a filter of $\Omega_L$.

PROOF: Tails$(L)$ is non empty family of subsets of $L$ and quasi basis and has non empty elements by [6] (22). □

Let $L$ be a non empty, transitive, reflexive relational structure. Assume $\Omega_L$ is directed. The functor TailsFilter$L$ yielding a filter of $\Omega_L$ is defined by the term

(Def. 15) $[\text{Tails}(L)]$.

Now we state the proposition:

(42) Let us consider a non empty, transitive, reflexive relational structure $L$. Suppose $\Omega_L$ is directed. Then Tails$(L)$ is a generalized basis of TailsFilter$L$. The theorem is a consequence of (22).

Let $L$ be a relational structure and $x$ be a family of subsets of $L$. The functor $\# x$ yielding a family of subsets of $\Omega_L$ is defined by the term

(Def. 16) $x$.

Now we state the proposition:

(43) Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, and a function $f$ from $\Omega_L$ into $X$. Suppose $\Omega_L$ is directed. Then $f^\circ(\# \text{Tails}(L))$ is a generalized basis of the image of filter TailsFilter$L$ under $f$. The theorem is a consequence of (42) and (37).

Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into $X$, and a subset $x$ of $X$. Now we state the propositions:

(44) Suppose $\Omega_L$ is directed and $x \in f^\circ(\# \text{Tails}(L))$. Then there exists an element $j$ of $L$ such that for every element $i$ of $L$ such that $i \geq j$ holds $f(i) \in x$.

(45) Suppose $\Omega_L$ is directed and there exists an element $j$ of $L$ such that for every element $i$ of $L$ such that $i \geq j$ holds $f(i) \in x$. Then there exists an element $b$ of Tails$(L)$ such that $f^\circ b \subseteq x$.

(46) Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into $X$, a filter $\mathcal{F}'$ of $X$, and a generalized basis $B$ of $\mathcal{F}'$. Suppose $\Omega_L$ is directed. Then $\mathcal{F}'$ is coarser than the image of filter TailsFilter$L$ under $f$ if and only if $B$ is coarser than $f^\circ(\# \text{Tails}(L))$. The theorem is a consequence of (43) and (33).

(47) Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into $X$, and a filter base $B$ of
X. Suppose \( \Omega_L \) is directed. Then \( B \) is coarser than \( f^0(\# \text{Tails}(L)) \) if and only if for every element \( b \) of \( B \), there exists an element \( i \) of \( L \) such that for every element \( j \) of \( L \) such that \( i \leq j \) holds \( f(j) \in b \). The theorem is a consequence of (44) and (45).

Let \( X \) be a non empty set and \( s \) be a sequence of \( X \). The elementary filter of \( s \) yielding a filter of \( X \) is defined by the term (Def. 17) the image of filter \( \text{FrechetFilter}(\mathbb{N}) \) under \( s \).

Now we state the propositions:

(48) There exists a sequence \( \mathcal{F}' \) of \( 2^\mathbb{N} \) such that for every element \( x \) of \( \mathbb{N} \), \( \mathcal{F}'(x) = \{ y, \text{where } y \text{ is an element of } \mathbb{N} : x \leq y \} \).

**Proof:** Define \( \mathcal{F}(\text{object}) = \{ y, \text{where } y \text{ is an element of } \mathbb{N} : \text{there exists an element } x_0 \text{ of } \mathbb{N} \text{ such that } x_0 = \$1 \text{ and } x_0 \leq y \} \). There exists a function \( f \) from \( \mathbb{N} \) into \( 2^\mathbb{N} \) such that for every object \( x \) such that \( x \in \mathbb{N} \) holds \( f(x) = \mathcal{F}(x) \). Consider \( \mathcal{F}' \) being a function from \( \mathbb{N} \) into \( 2^\mathbb{N} \) such that for every object \( x \) such that \( x \in \mathbb{N} \) holds \( \mathcal{F}'(x) = \mathcal{F}(x) \). For every element \( x \) of \( \mathbb{N} \), \( \mathcal{F}'(x) = \{ y, \text{where } y \text{ is an element of } \mathbb{N} : x \leq y \} \). □

(49) Let us consider a natural number \( n \). Then \( \mathbb{N} \setminus \{ t, \text{where } t \text{ is an element of } \mathbb{N} : n \leq t \} \) is finite.

**Proof:** \( \mathbb{N} \setminus \{ t, \text{where } t \text{ is an element of } \mathbb{N} : n \leq t \} \subseteq n + 1 \) by [8] (3), (5), [32] (4). □

(50) Let us consider an element \( p \) of the ordered \( \mathbb{N} \). Then \( \{ x, \text{where } x \text{ is an element of } \mathbb{N} : \text{there exists an element } p_0 \text{ of } \mathbb{N} \text{ such that } p = p_0 \text{ and } p_0 \leq x \} = \uparrow p \).

**Proof:** For every element \( p \) of the carrier of the ordered \( \mathbb{N} \), \( \{ x, \text{where } x \text{ is an element of the carrier of the ordered } \mathbb{N} : p \leq x \} = \uparrow p \) by [6] (18). □

Observe that \( \Omega \) the ordered \( \mathbb{N} \) is directed and the ordered \( \mathbb{N} \) is reflexive.

Now we state the proposition:

(51) Let us consider a denumerable set \( X \). Then \( \text{FrechetFilter}(X) = \) the set of all \( X \setminus A \) where \( A \) is a finite subset of \( X \).

Let us consider a sequence \( \mathcal{F}' \) of \( 2^\mathbb{N} \).

Let us assume that for every element \( x \) of \( \mathbb{N} \), \( \mathcal{F}'(x) = \{ y, \text{where } y \text{ is an element of } \mathbb{N} : x \leq y \} \). Now we state the propositions:

(52) \( \text{rng} \mathcal{F}' \) is a generalized basis of \( \text{FrechetFilter}(\mathbb{N}) \).

**Proof:** \( \text{FrechetFilter}(\mathbb{N}) = \) the set of all \( \mathbb{N} \setminus A \) where \( A \) is a finite subset of \( \mathbb{N} \). For every object \( t \) such that \( t \in \text{rng} \mathcal{F}' \) holds \( t \in \text{FrechetFilter}(\mathbb{N}) \). Reconsider \( \mathcal{F}_1 = \text{rng} \mathcal{F}' \) as a non empty subset of \( \text{FrechetFilter}(\mathbb{N}) \). \( \mathcal{F}_1 \) is filter basis by [21] (2), [4] (44), [11] (3). □
(53) $\# \text{Tails}(\text{the ordered } \mathbb{N}) = \text{rng } \mathcal{F}'$. The theorem is a consequence of (50).

Now we state the proposition:

(54) (i) $\# \text{Tails}(\text{the ordered } \mathbb{N})$ is a generalized basis of $\text{FrechetFilter}(\mathbb{N})$, and

(ii) $\text{TailsFilter} \text{ the ordered } \mathbb{N} = \text{FrechetFilter}(\mathbb{N})$.

The theorem is a consequence of (48), (53), (52), and (21).

The base of Frechet filter yielding a filter base of $\mathbb{N}$ is defined by the term

(Def. 18) $\# \text{Tails}(\text{the ordered } \mathbb{N})$.

Now we state the propositions:

(55) $\mathbb{N} \in \text{the base of Frechet filter}$.

(56) The base of Frechet filter is a generalized basis of $\text{FrechetFilter}(\mathbb{N})$.

(57) Let us consider a non empty set $X$, filters $\mathcal{F}_1, \mathcal{F}_2$ of $X$, and a filter $\mathcal{F}'$ of $X$. Suppose $\mathcal{F}'$ is finer than $\mathcal{F}_1$ and $\mathcal{F}'$ is finer than $\mathcal{F}_2$. Let us consider an element $M_1$ of $\mathcal{F}_1$, and an element $M_2$ of $\mathcal{F}_2$. Then $M_1 \cap M_2$ is not empty.

(58) Let us consider a non empty set $X$, and filters $\mathcal{F}_1, \mathcal{F}_2$ of $X$. Suppose for every element $M_1$ of $\mathcal{F}_1$ for every element $M_2$ of $\mathcal{F}_2$, $M_1 \cap M_2$ is not empty. Then there exists a filter $\mathcal{F}'$ of $X$ such that

(i) $\mathcal{F}'$ is finer than $\mathcal{F}_1$, and

(ii) $\mathcal{F}'$ is finer than $\mathcal{F}_2$.

Let $X$ be a set and $x$ be a subset of $X$. The functor $\text{SubsetToBooleSubset } x$ yielding an element of $2^X_{\subseteq}$ is defined by the term

(Def. 19) $x$.

Now we state the propositions:

(59) Let us consider an infinite set $X$. Then $X \in \text{the set of all } X \setminus A \text{ where } A \text{ is a finite subset of } X$.

(60) Let us consider a set $X$, and a subset $A$ of $X$. Then $\{B, \text{ where } B \text{ is an element of } 2^X_{\subseteq} : A \subseteq B\} = \{B, \text{ where } B \text{ is a subset of } X : A \subseteq B\}$.

(61) Let us consider a set $X$, and an element $a$ of $2^X_{\subseteq}$. Then $\uparrow a = \{Y, \text{ where } Y \text{ is a subset of } X : a \subseteq Y\}$.

(62) Let us consider a set $X$, and a subset $A$ of $X$. Then $\{B, \text{ where } B \text{ is an element of } 2^X_{\subseteq} : A \subseteq B\} = \uparrow \text{SubsetToBooleSubset } A$. The theorem is a consequence of (60).

(63) Let us consider a non empty set $X$, and a filter $\mathcal{F}'$ of $X$. Then $\bigcup \mathcal{F}' = X$.

(64) Let us consider an infinite set $X$. Then the set of all $X \setminus A \text{ where } A \text{ is a finite subset of } X$ is a filter of $X$. The theorem is a consequence of (59).
Let us consider a set $X$. Now we state the propositions:

65) $2^X$ is a filter of $2^X$.
66) $\{X\}$ is a filter of $2^X$.

67) Let us consider a non empty set $X$. Then $\{X\}$ is a filter of $X$.

Let us consider an element $A$ of $2^X$. Now we state the propositions:

68) $\{Y, \text{ where } Y \text{ is a subset of } X: A \subseteq Y\}$ is a filter of $2^X$.
69) $\{B, \text{ where } B \text{ is an element of } 2^X: A \subseteq B\}$ is a filter of $2^X$. The theorem is a consequence of (60) and (68).

Now we state the proposition:

70) Let us consider a non empty set $X$, and a non empty subset $B$ of $2^X$. Then for every elements $x, y$ of $B$, there exists an element $z$ of $B$ such that $z \subseteq x \cap y$ if and only if $B$ is filtered.

\textbf{Proof:} For every elements $x, y$ of $B$, there exists an element $z$ of $B$ such that $z \subseteq x \cap y$ by [19, (2)]. □

Let us consider a non empty set $X$ and a non empty subset $\mathcal{F}'$ of the lattice of subsets of $X$. Now we state the propositions:

71) $\mathcal{F}'$ is a filter of the lattice of subsets of $X$ if and only if for every elements $p, q$ of $\mathcal{F}'$, $p \cap q \in \mathcal{F}'$ and for every element $p$ of $\mathcal{F}'$ and for every element $q$ of the lattice of subsets of $X$ such that $p \subseteq q$ holds $q \in \mathcal{F}'$.
72) $\mathcal{F}'$ is a filter of the lattice of subsets of $X$ if and only if for every subsets $Y_1, Y_2$ of $X$, if $Y_1, Y_2 \in \mathcal{F}'$, then $Y_1 \cap Y_2 \in \mathcal{F}'$ and if $Y_1 \in \mathcal{F}'$ and $Y_1 \subseteq Y_2$, then $Y_2 \in \mathcal{F}'$. The theorem is a consequence of (71).

Now we state the propositions:

73) Let us consider a non empty set $X$, and a non empty family $\mathcal{F}$ of subsets of $X$. Suppose $\mathcal{F}$ is a filter of the lattice of subsets of $X$. Then $\mathcal{F}$ is a filter of $2^X$. The theorem is a consequence of (71).
74) Let us consider a non empty set $X$. Then every filter of $2^X$ is a filter of the lattice of subsets of $X$. The theorem is a consequence of (72).
75) Let us consider a non empty set $X$, and a non empty subset $\mathcal{F}'$ of the lattice of subsets of $X$. Then $\mathcal{F}'$ is filter of the lattice of subsets of $X$ and has non empty elements if and only if $\mathcal{F}'$ is a filter of $X$. The theorem is a consequence of (72).
76) Let us consider a non empty set $X$. Then every proper filter of $2^X$ is a filter of $X$.

\textbf{Proof:} $\mathcal{F}'$ has non empty elements by [19, (18)], [7, (4)]. □
77) Let us consider a non empty topological space $T$, and a point $x$ of $T$. Then the neighborhood system of $x$ is a filter of the carrier of $T$. 
Let $T$ be a non empty topological space and $\mathcal{F}'$ be a proper filter of $2^{\Omega_T}$. The functor $\text{BooleanFilterToFilter}(\mathcal{F}')$ yielding a filter of the carrier of $T$ is defined by the term

(Def. 20) $\mathcal{F}'$.  

Let $\mathcal{F}_1$ be a filter of the carrier of $T$ and $\mathcal{F}_2$ be a proper filter of $2^{\Omega_T}$. We say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ if and only if

(Def. 21) $\text{BooleanFilterToFilter}(\mathcal{F}_2) \subseteq \mathcal{F}_1$.

### 3. Limit of a Filter

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a filter of the carrier of $T$. The functor $\text{LimFilter}(\mathcal{F}')$ yielding a subset of $T$ is defined by the term

(Def. 22) $\{ x, \text{ where } x \text{ is a point of } T : \mathcal{F}' \text{ is finer than the neighborhood system of } x \}$.

Let $B$ be a filter base of the carrier of $T$. The functor $\text{LimB}$ yielding a subset of $T$ is defined by the term

(Def. 23) $\text{LimFilter([B])}$.

Now we state the proposition:

(78) Let us consider a non empty topological space $T$, and a filter $\mathcal{F}'$ of the carrier of $T$. Then there exists a proper filter $\mathcal{F}_1$ of $2^{\alpha}$ such that $\mathcal{F}' = \mathcal{F}_1$, where $\alpha$ is the carrier of $T$. The theorem is a consequence of (73) and (75).

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a filter of the carrier of $T$. The functor $\text{FilterToBooleanFilter}(\mathcal{F}', T)$ yielding a proper filter of $2^{\Omega_T}$ is defined by the term

(Def. 24) $\mathcal{F}'$.  

Let us consider a non empty topological space $T$, a point $x$ of $T$, and a filter $\mathcal{F}'$ of the carrier of $T$. Now we state the propositions:

(79) $x$ is a convergence point of $\mathcal{F}'$ and $T$ if and only if $x$ is a convergence point of $\text{FilterToBooleanFilter}(\mathcal{F}', T)$ and $T$.

(80) $x$ is a convergence point of $\mathcal{F}'$ and $T$ if and only if $x \in \text{LimFilter}(\mathcal{F}')$.

The theorem is a consequence of (78).

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a filter of $2^{\Omega_T}$. The functor $\text{LimFilterB}(\mathcal{F}')$ yielding a subset of $T$ is defined by the term

(Def. 25) $\{ x, \text{ where } x \text{ is a point of } T : \text{the neighborhood system of } x \subseteq \mathcal{F}' \}$.

Let us consider a non empty topological space $T$ and a filter $\mathcal{F}'$ of the carrier of $T$. Now we state the propositions:
(81) \( \text{LimFilter}(F') = \text{LimFilterB}(\text{FilterToBooleanFilter}(F', T)) \).

(82) \( \text{Lim}(\text{the net of } \text{FilterToBooleanFilter}(F', T)) = \text{LimFilter}(F') \).

(83) Let us consider a Hausdorff, non empty topological space \( T \), a filter \( F' \) of the carrier of \( T \), and points \( p, q \) of \( T \). If \( p, q \in \text{LimFilter}(F') \), then \( p = q \).

Let \( T \) be a Hausdorff, non empty topological space and \( F' \) be a filter of the carrier of \( T \). Note that \( \text{LimFilter}(F') \) is trivial.

Let \( X \) be a non empty set, \( T \) be a non empty topological space, \( f \) be a function from \( X \) into the carrier of \( T \), and \( F' \) be a filter of \( X \). The functor \( \text{lim}_{\mathcal{F}'} f \) yielding a subset of \( \Omega_T \) is defined by the term

(Def. 26) \( \text{LimFilter}(\text{the image of filter } F' \text{ under } f) \).

Let \( L \) be a non empty, transitive, reflexive relational structure and \( f \) be a function from \( \Omega_L \) into the carrier of \( T \). The functor \( \text{LimF}(f) \) yielding a subset of \( \Omega_T \) is defined by the term

(Def. 27) \( \text{LimFilter}(\text{the image of filter TailsFilter } L \text{ under } f) \).

Now we state the proposition:

(84) Let us consider a non empty topological space \( T \), a non empty, transitive, reflexive relational structure \( L \), a function \( f \) from \( \Omega_L \) into the carrier of \( T \), a point \( x \) of \( T \), and a generalized basis \( B \) of \( \text{BooleanFilterToFilter}(\text{the neighborhood system of } x) \). Suppose \( \Omega_L \) is directed. Then \( x \in \text{LimF}(f) \) if and only if for every element \( b \) of \( B \), there exists an element \( i \) of \( L \) such that for every element \( j \) of \( L \) such that \( i \leq j \) holds \( f(j) \in b \). The theorem is a consequence of (46), (29), and (47).

Let \( T \) be a non empty topological space and \( s \) be a sequence of \( T \). The functor \( \text{LimF}(s) \) yielding a subset of \( T \) is defined by the term

(Def. 28) \( \text{LimFilter}(\text{the elementary filter of } s) \).

Now we state the proposition:

(85) Let us consider a non empty topological space \( T \), and a sequence \( s \) of \( T \). Then \( \lim_{\text{FrechetFilter}(\mathbb{N})} s = \text{LimF}(s) \).

Let us consider a non empty topological space \( T \) and a point \( x \) of \( T \).

(86) The neighborhood system of \( x \) is a filter base of \( \Omega_T \). The theorem is a consequence of (76), (13), and (29).

(87) Every generalized basis of \( \text{BooleanFilterToFilter}(\text{the neighborhood system of } x) \) is a filter base of \( \Omega_T \).

(88) Let us consider a non empty set \( X \), a sequence \( s \) of \( X \), and a filter base \( B \) of \( X \). Then \( B \) is coarser than \( s^\circ \) (the base of Frechet filter) if and only if for every element \( b \) of \( B \), there exists an element \( i \) of the ordered \( \mathbb{N} \) such that for every element \( j \) of the ordered \( \mathbb{N} \) such that \( i \leq j \) holds \( s(j) \in b \).
(89) Let us consider a non empty topological space $T$, a sequence $s$ of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Then $x \in \lim_{n \to \infty} s \circ q$ if and only if $B$ is coarser than $s^\circ$ (the base of Frechet filter). The theorem is a consequence of (46) and (54).

(90) Let us consider a non empty topological space $T$, a sequence $s$ of $\Omega_T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Then $B$ is coarser than $s \circ q$ (the base of Frechet filter) if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leq j$ holds $s(j) \in b$. The theorem is a consequence of (29) and (47).

Let us consider a non empty topological space $T$, a sequence $s$ of the carrier of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$).

(91) $x \in \lim_{n \to \infty} s \circ q$ if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leq j$ holds $s(j) \in b$. The theorem is a consequence of (89) and (90).

(92) $x \in \lim_{n \to \infty} s$ if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leq j$ holds $s(j) \in b$. The theorem is a consequence of (91).

4. Nets

Let $L$ be a 1-sorted structure and $s$ be a sequence of the carrier of $L$. The net of $s$ yielding a non empty, strict net structure over $L$ is defined by the term

(Def. 29) $\langle \mathbb{N}, \leq \mathbb{N}, s \rangle$.

Let $L$ be a non empty 1-sorted structure. Let us note that the net of $s$ is non empty.

Now we state the proposition:

(93) Let us consider a non empty 1-sorted structure $L$, a set $B$, and a sequence $s$ of the carrier of $L$. Then the net of $s$ is eventually in $B$ if and only if there exists an element $i$ of the net of $s$ such that for every element $j$ of the net of $s$ such that $i \leq j$ holds (the net of $s)(j) \in B$.

Let us consider a non empty topological space $T$, a sequence $s$ of the carrier of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Now we state the propositions:

(94) for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leq j$ holds $s(j) \in b$ if
and only if for every element $b$ of $B$, there exists an element $i$ of the net of $s$ such that for every element $j$ of the net of $s$ such that $i \leq j$ holds (the net of $s)(j) \in b$.

(95) \( x \in \text{LimF}(s) \) if and only if for every element $b$ of $B$, the net of $s$ is eventually in $b$. The theorem is a consequence of (92), (94), and (93).

(96) \( x \in \text{LimF}(s) \) if and only if for every element $b$ of $B$, there exists an element $i$ of $\mathbb{N}$ such that for every element $j$ of $\mathbb{N}$ such that $i \leq j$ holds \( s(j) \in b \). The theorem is a consequence of (91).

(97) \( x \in \text{LimF}(s) \) if and only if for every element $b$ of $B$, there exists a natural number $i$ such that for every natural number $j$ such that $i \leq j$ holds \( s(j) \in b \). The theorem is a consequence of (96).

**REFERENCES**

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