Polynomially Bounded Sequences and Polynomial Sequences

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Summary. In this article, we formalize polynomially bounded sequences that plays an important role in computational complexity theory. Class P is a fundamental computational complexity class that contains all polynomial-time decision problems [11], [12]. It takes polynomially bounded amount of computation time to solve polynomial-time decision problems by the deterministic Turing machine. Moreover we formalize polynomial sequences [5].

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The notation and terminology used in this paper have been introduced in the following articles: [26], [18], [16], [17], [6], [22], [10], [7], [8], [24], [14], [1], [2], [3], [13], [20], [27], [28], [21], [25], and [9].

1. Preliminaries

Now we state the proposition:

(1) Let us consider natural numbers $m$, $k$. If $1 \leq m$, then $1 \leq m^k$.

Let us consider natural numbers $m$, $n$. Now we state the propositions:

(2) $m \leq m^{n+1}$.

(3) If $2 \leq m$, then $n + 1 \leq m^n$. 
(4) Let us consider a natural number $k$. Then $2 \cdot k \leq 2^k$.

**Proof:** Define $P(n) \equiv 2 \cdot \frac{1}{k} \leq 2 \cdot \frac{1}{k + 1}$. For every natural number $n$ such that $P(n)$ holds $P(n + 1)$ by [20] (25), [24] (5), [1] (14), (2). For every natural number $n$, $P[n]$ from [1] Sch. 2. □

(5) Let us consider natural numbers $k$, $n$. If $k \leq n$, then $n + k \leq 2^n$.

**Proof:** Define $P(n) \equiv 1 + k + \frac{k}{n} \leq 2 + k$. For every natural number $n$ such that $P(n)$ holds $P(n + 1)$ by [20] (27), (25), (24). For every natural number $n$, $P[n]$ from [1] Sch. 2. □

(6) Let us consider natural numbers $k$, $m$. If $2 \cdot k + 1 \leq m$, then $2^k \leq 2^m / m$.

The theorem is a consequence of (5).

(7) Let us consider real numbers $a$, $b$, $c$. If $1 < a$ and $0 < b \leq c$, then

$$\log_a b \leq \log_a c.$$ 

Let us consider a natural number $n$ and a real number $a$. Now we state the propositions:

(8) If $1 < a$, then $a^n < a^{n+1}$.

(9) If $1 \leq a$, then $a^n \leq a^{n+1}$.

(10) There exists a partial function $g$ from $\mathbb{R}$ to $\mathbb{R}$ such that

(i) $\text{dom } g = ]0, +\infty[$, and

(ii) for every real number $x$ such that $x \in ]0, +\infty[$ holds $g(x) = \log_2 x$, and

(iii) $g$ is differentiable on $]0, +\infty[$, and

(iv) for every real number $x$ such that $x \in ]0, +\infty[$ holds $g$ is differentiable in $x$ and $g'(x) = \log_2 e / x$ and $0 < g'(x)$.

**Proof:** Set $g = \log_2 e \cdot (\text{the function ln}).$ For every real number $d$ such that $d \in ]0, +\infty[$ holds $g(d) = \log_2 d$ by [20] (56). For every real number $x$ such that $x \in ]0, +\infty[$ holds $g$ is differentiable in $x$ and $g'(x) = \log_2 e / x$ and $0 < g'(x)$ by [23] (18), [22] (15), [20] (57), [23] (11). □

(11) There exists a partial function $f$ from $\mathbb{R}$ to $\mathbb{R}$ such that

(i) $]e, +\infty[ = \text{dom } f$, and

(ii) for every real number $x$ such that $x \in \text{dom } f$ holds $f(x) = x / \log_2 x$, and

(iii) $f$ is differentiable on $]e, +\infty[$, and

(iv) for every real number $x_0$ such that $x_0 \in ]e, +\infty[$ holds $0 \leq f'(x_0)$, and

(v) $f$ is non-decreasing.
Proof: Consider \( g \) being a partial function from \( \mathbb{R} \) to \( \mathbb{R} \) such that \( \text{dom} \ g = [0, +\infty[ \) and for every real number \( x \) such that \( x \in ]0, +\infty[ \) holds \( g(x) = \log_2 x \) and \( g \) is differentiable on \( ]0, +\infty[ \) and for every real number \( x \) such that \( x \in ]0, +\infty[ \) holds \( g(x) = \log_2 e/x \) and \( 0 < g'(x) \). Set \( g_0 = g|_e, +\infty[ \). For every object \( x \) such that \( x \in ]e, +\infty[ \) holds \( x \in ]0, +\infty[ \) by \([23, (11)]\). Set \( f = \text{id}_{\Omega_2/g_0}. g_0^{-1}(\{0\}) = \emptyset \) by \([23, (11)]\), \([7, (49)]\), \([4, (10)]\), \([20, (52)]\). For every real number \( x \) such that \( x \in ]e, +\infty[ \) holds \( f(x) = x/\log_2 x \) by \([7, (49)]\). For every real number \( x \) such that \( x \in ]e, +\infty[ \) holds \( f \) is differentiable in \( x \) and \( f'(x) = \log_2 x - \log_2 e/(\log_2 x)^2 \) by \([23, (11)]\), \([7, (49)]\), \([4, (10)]\), \([20, (52)]\). For every real number \( x \) such that \( x \in ]e, +\infty[ \) holds \( 0 \leq f'(x) \) by \([20, (57)]\), \([23, (11)]\).

(12) Let us consider real numbers \( x, y \). If \( e < x \leq y \), then \( x/\log_2 x \leq y/\log_2 y \).

The theorem is a consequence of (11).

(13) Let us consider a natural number \( k \). Suppose \( e < k \). Then there exists a natural number \( N \) such that for every natural number \( n \) such that \( N \leq n \) holds \( 2^k \leq n/\log_2 n \). The theorem is a consequence of (12) and (6).

Let us consider a natural number \( x \). Let us assume that \( 1 < x \).

(14) There exists a natural number \( N \) such that for every natural number \( n \) such that \( N \leq n \) holds \( 4 < n/\log_2 n \).

(15) There exist natural numbers \( N, c \) such that for every natural number \( n \) such that \( N \leq n \) holds \( n^x \leq c \cdot x^n \).

(16) Let us consider a natural number \( x \). Suppose \( 1 < x \). Then there exist no natural numbers \( N, c \) such that for every natural number \( n \) such that \( N \leq n \) holds \( 2^n \leq c \cdot n^x \).

Proof: Consider \( N \) being a natural number such that there exists a natural number \( c \) such that for every natural number \( n \) such that \( N \leq n \) holds \( 2^n \leq c \cdot n^x \). \( N \neq 0 \) by \([20, (42), (24)]\). Consider \( c \) being a natural number such that for every natural number \( n \) such that \( N \leq n \) holds \( 2^n \leq c \cdot n^x \). There exists an element \( n \) of \( \mathbb{N} \) such that \( N \leq n \) and \( 0 < n - (x/4) \) by \([24, (6), (3)]\). Consider \( n \) being an element of \( \mathbb{N} \) such that \( N \leq n \) and \( 0 < n - (x/4) \). \( 0 < c \) by \([20, (34)]\). For every natural number \( k \) such that \( 1 \leq k \) holds \( 2^k \cdot n \leq c \cdot (k \cdot n)^x \). For every natural number \( k \) such that \( 1 \leq k \) holds \( k \cdot n \leq \log_2 c + x \cdot \log_2 k + x \cdot \log_2 n \) by \([20, (34)]\), \([7, (55), (52), (53)]\). Consider \( Z \) being an element of \( \mathbb{N} \) such that for every natural number \( k \) such that \( Z \leq k \) holds \( 4 < k/\log_2 k \). There exists a natural number \( k \) such that \( Z \leq k \) and \( \log_2 c + x \cdot \log_2 n/\log_2 n - (x/4) \leq k \) by \([24, (6), (3)]\). There exists a natural number \( k \) such that \( Z \leq k \) and \( \log_2 c + x \cdot \log_2 n/\log_2 n - (x/4) \leq k \) and \( 1 < k \) by \([1, (11)]\). Consider \( k \) being a natural number such that \( Z \leq k \) and \( 1 < k \) and \( \log_2 c + x \cdot \log_2 n/\log_2 n - (x/4) \leq k \). \( \square \)
(17) Let us consider natural numbers $a, b$. If $a \leq b$, then $\{n^a\}_{n \in \mathbb{N}} \in O(\{n^b\}_{n \in \mathbb{N}})$.

(18) Let us consider a natural number $x$. Suppose $1 < x$. Then there exist no natural numbers $N, c$ such that for every natural number $n$ such that $N \leq n$ holds $x^n \leq c \cdot n^x$.

Proof: There exist natural numbers $N, c$ such that for every natural number $n$ such that $N \leq n$ holds $2^n \leq c \cdot n^x$ by [24, (7)]. □

(19) Let us consider a non-negative real number $a$, and a natural number $n$. If $1 \leq a$, then $0 < \{n^a\}_{n \in \mathbb{N}}(n)$.

2. POLYNOMIALLY BOUNDED SEQUENCES

Let $p$ be a sequence of real numbers. We say that $p$ is polynomially bounded if and only if

(Def. 1) there exists a natural number $k$ such that $p \in O(\{n^k\}_{n \in \mathbb{N}})$.

Now we state the propositions:

(20) Let us consider a sequence $f$ of real numbers. Suppose $f$ is not polynomially bounded. Let us consider a natural number $k$. Then $f \notin O(\{n^k\}_{n \in \mathbb{N}})$.

(21) Let us consider a sequence $f$ of real numbers. Suppose for every natural number $k$, $f \notin O(\{n^k\}_{n \in \mathbb{N}})$. Then $f$ is not polynomially bounded.

(22) Let us consider a positive real number $a$. Then $\{a^{1-n+0}\}_{n \in \mathbb{N}}$ is positive.

Let us consider a real number $a$. Now we state the propositions:

(23) If $1 \leq a$, then $\{a^{1-n+0}\}_{n \in \mathbb{N}}$ is non-decreasing. The theorem is a consequence of (9).

(24) If $1 < a$, then $\{a^{1-n+0}\}_{n \in \mathbb{N}}$ is increasing. The theorem is a consequence of (8).

(25) Let us consider a natural number $a$. If $1 < a$, then $\{a^{1-n+0}\}_{n \in \mathbb{N}}$ is not polynomially bounded.

Proof: Consider $k$ being a natural number such that $\{a^{1-n+0}\}_{n \in \mathbb{N}} \in O(\{n^k\}_{n \in \mathbb{N}})$. Reconsider $f = \{n^k\}_{n \in \mathbb{N}}$ as an eventually positive sequence of real numbers. Reconsider $t = \{a^{1-n+0}\}_{n \in \mathbb{N}}$ as an eventually non-negative sequence of real numbers. $t \in O(f)$ and for every element $n$ of $\mathbb{N}$ such that $1 \leq n$ holds $0 < f(n)$. Consider $c$ being a real number such that $c > 0$ and for every element $n$ of $\mathbb{N}$ such that $n \geq 1$ holds $\{a^{1-n+0}\}_{n \in \mathbb{N}}(n) \leq c \cdot \{n^k\}_{n \in \mathbb{N}}(n)$. For every natural number $n$ such that $n \geq 1$ holds $2^n \leq c \cdot n^k$ by [24, (7)]. There exist natural numbers $N, b$ such that for every natural number $n$ such that $N \leq n$ holds $2^n \leq b \cdot n^k$ by [24, (3)]. □
3. Polynomial Sequences

Now we state the proposition:

(26) Let us consider a finite 0-sequence \( x \) of \( \mathbb{R} \), and a sequence \( y \) of real numbers. Then

(i) \( x \cdot y \) is a finite transfinite sequence of elements of \( \mathbb{R} \), and

(ii) \( \text{dom}(x \cdot y) = \text{dom} x \), and

(iii) for every object \( i \) such that \( i \in \text{dom} x \) holds \( (x \cdot y)(i) = x(i) \cdot y(i) \).

Let \( x \) be a finite 0-sequence of \( \mathbb{R} \) and \( y \) be a sequence of real numbers. Observe that the functor \( x \cdot y \) yields a finite 0-sequence of \( \mathbb{R} \). Now we state the proposition:

(27) Let us consider a finite 0-sequence \( d \) of \( \mathbb{R} \), and natural numbers \( x, i \).

Suppose \( i \in \text{dom} d \). Then \( (d \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}})(i) = d(i) \cdot x^i \). The theorem is a consequence of (26).

Let \( c \) be a finite 0-sequence of \( \mathbb{R} \). The functor \( \text{Seq}_{\text{poly}}(c) \) yielding a sequence of real numbers is defined by

(Def. 2) for every natural number \( x \), \( \text{it}(x) = \sum(c \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}}) \).

Let us consider a finite 0-sequence \( d \) of \( \mathbb{R} \) and a natural number \( k \). Now we state the propositions:

(28) Suppose \( \text{len} d = k + 1 \). Then there exists a real number \( a \) and there exists a finite 0-sequence \( d_1 \) of \( \mathbb{R} \) and there exists a sequence \( y \) of real numbers such that \( \text{len} d_1 = k \) and \( d_1 = d \upharpoonright k \) and \( a = d(k) \) and \( d = d_1 \upharpoonright \langle a \rangle \) and \( \text{Seq}_{\text{poly}}(d) = \text{Seq}_{\text{poly}}(d_1) + y \) and for every natural number \( i \), \( y(i) = a \cdot i^k \).

PROOF: Consider \( a \) being a real number, \( d_1 \) being a finite 0-sequence of \( \mathbb{R} \) such that \( \text{len} d_1 = k \) and \( d_1 = d \upharpoonright k \) and \( a = d(k) \) and \( d = d_1 \upharpoonright \langle a \rangle \). Define \( \mathcal{F} \langle \text{natural number} \rangle = a \cdot \cdot k \). Consider \( y \) being a sequence of real numbers such that for every natural number \( x \), \( y(x) = \mathcal{F}(x) \) from [15, Sch. 1]. For every element \( x \) of \( \mathbb{N} \), \( (\text{Seq}_{\text{poly}}(d))(x) = (\text{Seq}_{\text{poly}}(d_1) + y)(x) \) by (26), [11 (13), (44)], (27). □

(29) If \( \text{len} d = 1 \), then there exists a real number \( a \) such that \( a = d(0) \) and for every natural number \( x \), \( (\text{Seq}_{\text{poly}}(d))(x) = a \). The theorem is a consequence of (26).

(30) If \( \text{len} d = 1 \) and \( d \) is non-negative yielding, then \( \text{Seq}_{\text{poly}}(d) \in O(\{n^k\}_{n \in \mathbb{N}}) \). The theorem is a consequence of (29).

(31) Let us consider a natural number \( k \), a real number \( a \), and a sequence \( y \) of real numbers. Suppose \( 0 \leq a \) and for every natural number \( i \), \( y(i) = a \cdot i^k \). Then \( y \in O(\{n^k\}_{n \in \mathbb{N}}) \).
(32) Let us consider natural numbers \( k, n \). If \( k \leq n \), then \( O(\{n^k\}_{n \in \mathbb{N}}) \subseteq O(\{n^n\}_{n \in \mathbb{N}}) \).

**Proof:** Consider \( i \) being a natural number such that \( n = k + i \). Define \( \mathcal{P}[\text{natural number}] \equiv O(\{n^k\}_{n \in \mathbb{N}}) \subseteq O(\{n^{(k+i)}\}_{n \in \mathbb{N}}) \). For every natural number \( x \) such that \( \mathcal{P}[x] \) holds \( \mathcal{P}[x+1] \). For every natural number \( x \), \( \mathcal{P}[x] \) from \([1\text{ Sch. 2}]. \)

(33) Let us consider a natural number \( k \), and a non-negative yielding finite 0-sequence \( c \) of \( \mathbb{R} \). Suppose \( \text{len } c = k + 1 \). Then \( \text{Seq}_{\text{poly}}(c) \in O(\{\{n^k\}_{n \in \mathbb{N}} \). 

**Proof:** Define \( \mathcal{P}[\text{natural number}] \equiv O(\{n^k\}_{n \in \mathbb{N}}) \subseteq O(\{n^{\text{len } c+1}\}_{n \in \mathbb{N}}) \). For every natural number \( k \) such that \( \mathcal{P}[k] \) holds \( \mathcal{P}[k+1] \) by \([7\text{ (47)}],[1\text{ (13), (39)}]\). For every natural number \( k \), \( \mathcal{P}[k] \) from \([1\text{ Sch. 2}]. \)

(34) Let us consider a natural number \( k \), and a finite 0-sequence \( c \) of \( \mathbb{R} \). Then there exists a finite 0-sequence \( d \) of \( \mathbb{R} \) such that

(i) \( \text{len } d = \text{len } c \), and

(ii) for every natural number \( i \) such that \( i \in \text{dom } d \) holds \( d(i) = |c(i)| \).

**Proof:** Define \( \mathcal{F}(\text{natural number}) = |c(\{\text{len } c\}((\in \mathbb{R}) \). Consider \( d \) being a finite 0-sequence of \( \mathbb{R} \) such that \( \text{len } d = \text{len } c \) and for every natural number \( j \) such that \( j \in \text{len } c \) holds \( d(j) = \mathcal{F}(j) \) from \([18\text{ Sch. 1}]. \)

(35) Let us consider a finite 0-sequence \( c \) of \( \mathbb{R} \), and a finite 0-sequence \( d \) of \( \mathbb{R} \). Suppose \( \text{len } d = \text{len } c \) and for every natural number \( i \) such that \( i \in \text{dom } d \) holds \( d(i) = |c(i)| \). Let us consider a natural number \( n \). Then \( (\text{Seq}_{\text{poly}}(c))(n) \leq (\text{Seq}_{\text{poly}}(d))(n) \).

**Proof:** \( \text{dom}(d \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}}) = \text{dom } d \). For every natural number \( i \) such that \( i \in \text{dom}(c \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}}) \) holds \( (c \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}})(i) \leq (d \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}})(i) \) by \((26), (27), [19\text{ (4)}]. \)

(36) Let us consider a natural number \( k \), and a 0-sequence \( c \) of \( \mathbb{R} \). Suppose \( \text{len } c = k + 1 \) and \( \text{Seq}_{\text{poly}}(c) \) is eventually nonnegative. Then \( \text{Seq}_{\text{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}}) \).

**Proof:** Consider \( d \) being a finite 0-sequence of \( \mathbb{R} \) such that \( \text{len } d = \text{len } c \) and for every natural number \( i \) such that \( i \in \text{dom } d \) holds \( d(i) = |c(i)| \). For every natural number \( i \) such that \( i \in \text{dom } d \) holds \( 0 < d(i) \) by \([6\text{ (46)}]. \)

For every real number \( r \) such that \( r \in \text{rng } d \) holds \( 0 < r \). \( \text{Seq}_{\text{poly}}(d) \in O(\{n^k\}_{n \in \mathbb{N}}) \). Consider \( t \) being an element of \( \mathbb{R}^\mathbb{N} \) such that \( \text{Seq}_{\text{poly}}(d) = t \) and there exists a real number \( c \) and there exists an element \( N \) of \( \mathbb{N} \) such that \( c > 0 \) and for every element \( n \) of \( \mathbb{N} \) such that \( n \geq N \) holds \( t(n) \leq c \cdot \{n^k\}_{n \in \mathbb{N}}(n) \) and \( t(n) \geq 0 \). Consider \( N_1 \) being a natural number such that for every natural number \( n \) such that \( N_1 \leq n \) holds \( 0 \leq (\text{Seq}_{\text{poly}}(c))(n) \).
Consider a real number, $N_2$ being an element of $\mathbb{N}$ such that $a > 0$ and for every element $n$ of $\mathbb{N}$ such that $n \geq N_2$ holds $t(n) \leq a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $t(n) \geq 0$. Set $N = N_1 + N_2$. For every element $n$ of $\mathbb{N}$ such that $n \geq N$ holds $(\text{Seq}_{\text{poly}}(c))(n) \leq a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $(\text{Seq}_{\text{poly}}(c))(n) \geq 0$ by $[\Pi] (11)$, (35). □

(37) Let us consider natural numbers $k$, $n$. If $0 < n$, then $n \cdot \{n^k\}_{n \in \mathbb{N}}(n) = \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$.

(38) Let us consider a finite 0-sequence $c$ of $\mathbb{R}$. Suppose $\text{len} \ c = 0$. Let us consider a natural number $x$. Then $(\text{Seq}_{\text{poly}}(c))(x) = 0$.

(39) Let us consider an eventually nonnegative sequence $f$ of real numbers, and a natural number $k$. Suppose $f \in O(\{n^k\}_{n \in \mathbb{N}})$. Then there exists a natural number $N$ such that for every natural number $n$ such that $N \leq n$ holds $f(n) \leq \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$. The theorem is a consequence of (37).

(40) Let us consider a finite 0-sequence $c$ of $\mathbb{R}$. Then there exists a finite 0-sequence $a_1$ of $\mathbb{R}$ such that

(i) $a_1 = |c|$, and

(ii) for every natural number $n$, $(\text{Seq}_{\text{poly}}(c))(n) \leq (\text{Seq}_{\text{poly}}(a_1))(n)$.

**Proof:** Reconsider $a_1 = |c|$ as a finite 0-sequence of $\mathbb{R}$. Set $m_1 = c \cdot \{n^{1-n+0}\}_{n \in \mathbb{N}}$. Set $m_2 = a_1 \cdot \{n^{1-n+0}\}_{n \in \mathbb{N}}$. For every natural number $x$ such that $x \in \text{dom} m_1$ holds $m_1(x) \leq m_2(x)$ by $[\Pi], (4)$.

(41) Let us consider finite 0-sequences $c$, $a_1$ of $\mathbb{R}$. Suppose $a_1 = |c|$. Let us consider a natural number $n$. Then $|(\text{Seq}_{\text{poly}}(c))(n)| \leq (\text{Seq}_{\text{poly}}(a_1))(n)$.

**Proof:** Define $\mathcal{P}[:\text{natural number}] \equiv$ for every finite 0-sequences $c$, $a_1$ of $\mathbb{R}$ such that $\text{len} c = 1$ and $a_1 = |c|$ for every natural number $x$, $|\text{Seq}_{\text{poly}}(c))(x)| \leq (\text{Seq}_{\text{poly}}(a_1))(x)$. $\mathcal{P}[0]$ by (26), $[6], (44)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (28), $[7], (47), [15], (7)$, $[6], (56), (65)$. For every natural number $n$, $\mathcal{P}[n]$ from $[\Pi]$, Sch. 2.

(42) Let us consider a real number $a$. Suppose $0 < a$. Let us consider a natural number $k$, and a non-negative yielding finite 0-sequence $d$ of $\mathbb{R}$. Suppose $\text{len} d = k$. Then there exists a natural number $N$ such that for every natural number $x$ such that $N \leq x$ for every natural number $i$ such that $i \in \text{dom} d$ holds $d(i) \cdot x^i \cdot k \leq a \cdot x^k$.

**Proof:** For every natural number $i$ such that $i \in \text{dom} d$ holds $0 \leq d(i)$ by $[\Pi], (3)$.

(43) Let us consider a natural number $k$, a finite 0-sequence $d$ of $\mathbb{R}$, a real number $a$, and a sequence $y$ of real numbers. Suppose $0 < a$ and $\text{len} d = k$ and for every natural number $x$, $y(x) = a \cdot x^k$. Then there exists a natural number $N$ such that for every natural number $x$ such that $N \leq x$ holds...
\(|\text{Seq}_{\text{poly}}(d)(x)| \leq y(x)\). The theorem is a consequence of (38), (42), (26), (27), and (41).

\((44)\) Let us consider a natural number \(k\), and a finite 0-sequence \(d\) of \(\mathbb{R}\). Suppose \(\text{len} \ d = k+1\) and \(0 < d(k)\). Then \(\text{Seq}_{\text{poly}}(d)\) is eventually nonnegative.

Proof: Consider \(a\) being a real number, \(d_1\) being a finite 0-sequence of \(\mathbb{R}\), \(y\) being a sequence of real numbers such that \(\text{len} \ d_1 = k\) and \(d_1 = d|k\) and \(a = d(k)\) and \(d = d_1 \cap \langle a \rangle\) and \(\text{Seq}_{\text{poly}}(d) = \text{Seq}_{\text{poly}}(d_1) + y\) and for every natural number \(i\), \(y(i) = a \cdot i^k\). Consider \(N\) being a natural number such that for every natural number \(i\) such that \(N \leq i\) holds \(|(\text{Seq}_{\text{poly}}(d_1))(i)| \leq y(i)\). For every natural number \(i\) such that \(N \leq i\) holds \(0 \leq (\text{Seq}_{\text{poly}}(d))(i)\) by [19, (4)], [15, (7)]. □

Let us consider a natural number \(k\) and a finite 0-sequence \(c\) of \(\mathbb{R}\).

Let us assume that \(\text{len} \ c = k+1\) and \(0 < c(k)\). Now we state the propositions:

\((45)\) \(\text{Seq}_{\text{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})\).

\((46)\) \(\text{Seq}_{\text{poly}}(c)\) is polynomially bounded. The theorem is a consequence of (36) and (44).

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References


Polynomially bounded sequences and polynomial sequences


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