Morley’s Trisector Theorem

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Summary. Morley’s trisector theorem states that “The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle” [10].

There are many proofs of Morley’s trisector theorem [12, 16, 9, 13, 8, 20, 3, 18]. We follow the proof given by A. Letac in [15].

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The notation and terminology used in this paper have been introduced in the following articles: [11], [7], [14], [19], [2], [4], [23], [5], [24], [21], [22], and [6].

1. Preliminaries

From now on on A, B, C, D, E, F, G denote points of E_2.

Now we state the propositions:

(1) \( \angle(A, B, A) = 0 \).

(2) \( 0 \leq \angle(A, B, C) < 2 \cdot \pi \).

(3) (i) \( 0 \leq \angle(A, B, C) < \pi \), or

(ii) \( \angle(A, B, C) = \pi \), or

(iii) \( \pi < \angle(A, B, C) < 2 \cdot \pi \).

The theorem is a consequence of (2).

(4) \( |F - E|^2 = |A - E|^2 + |A - F|^2 - 2 \cdot |A - E| \cdot |A - F| \cdot \cos \angle(E, A, F) \).

(5) If A, B, C are mutually different and \( 0 < \angle(A, B, C) < \pi \), then \( 0 < \angle(B, C, A) < \pi \) and \( 0 < \angle(C, A, B) < \pi \).
(6) Suppose $A$, $B$, $C$ are mutually different and $\angle(A, B, C) = 0$. Then
(i) $\angle(B, C, A) = 0$ and $\angle(C, A, B) = \pi$, or
(ii) $\angle(B, C, A) = \pi$ and $\angle(C, A, B) = 0$ and $\angle(A, B, C) + \angle(B, C, A) + \angle(C, A, B) = \pi$.
(7) Suppose $A$, $B$, $C$ are mutually different and $\angle(A, B, C) = \pi$. Then
(i) $\angle(B, C, A) = 0$, and
(ii) $\angle(C, A, B) = 0$, and
(iii) $\angle(A, B, C) + \angle(B, C, A) + \angle(C, A, B) = \pi$.
(8) If $A$, $B$, $C$ are mutually different and $\angle(A, B, C) > \pi$, then $\angle(A, B, C) + \angle(B, C, A) + \angle(C, A, B) = 5 \cdot \pi$.
Let us assume that $\angle(C, B, A) < \pi$. Now we state the propositions:
(9) $0 \leq \text{area of } \triangle(A, B, C)$. The theorem is a consequence of (2).
(10) $0 \leq \varnothing(A, B, C)$. The theorem is a consequence of (9).

2. Morley’s Theorem

Now we state the propositions:
(11) Suppose $A$, $F$, $C$ form a triangle and $\angle(C, F, A) < \pi$ and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$ and $\angle(A, C, B)/3 + \angle(B, C, A)/3 = \pi/3$. Then $|\triangle A - F| \cdot \sin((\pi/3) - (\angle(C, B, A)/3)) = |\triangle A - C| \cdot \sin(\angle(A, C, B)/3)$.
(12) Suppose $A$, $B$, $C$ form a triangle and $\angle(C, F, A) < \pi$ and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$ and $\angle(A, C, B)/3 + \angle(B, C, A)/3 + \angle(C, B, A)/3 = \pi/3$ and $\sin((\pi/3) - (\angle(C, B, A)/3)) \neq 0$. Then $|\triangle A - F| = 4 \cdot \varnothing(A, B, C) \cdot \sin(\angle(C, B, A)/3) \cdot \sin((\pi/3) + (\angle(C, B, A)/3)) \cdot \sin(\angle(A, C, B)/3)$. The theorem is a consequence of (11).
(13) Suppose $C$, $A$, $B$ form a triangle and $A$, $F$, $C$ form a triangle and $F$, $A$, $E$ form a triangle and $E$, $A$, $B$ form a triangle and $\angle(B, A, E) = \angle(B, A, C)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$. Then $\angle(E, A, F) = \angle(B, A, C)/3$. PROOF: $\angle(E, A, F) \neq 4 \cdot \pi + (\angle(B, A, C)/3)$ by [17] (5), (2), [7] (30). $\angle(E, A, F) \neq 2 \cdot \pi + (\angle(B, A, C)/3)$ by (2), [7] (30). □
(14) Suppose $C$, $A$, $B$ form a triangle and $\angle(A, C, B) < \pi$ and $A$, $F$, $C$ form a triangle and $F$, $A$, $E$ form a triangle and $E$, $A$, $B$ form a triangle and $\angle(B, A, E) = \angle(B, A, C)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$. Then $(\pi/3) + (\angle(A, C, B)/3) + (\pi/3) + (\angle(C, B, A)/3)) + \angle(E, A, F) = \pi$. The theorem is a consequence of (13).
If $A$, $C$, $B$ form a triangle, then $\sin((\pi/3) - (\angle(A, C, B)/3)) \neq 0$. The theorem is a consequence of (2).

Suppose $A$, $B$, $C$ form a triangle and $A$, $B$, $E$ form a triangle and $\angle(E, B, A) = \angle(C, B, A)/3$ and $\angle(B, A, E) = \angle(B, A, C)/3$ and $A$, $F$, $C$ form a triangle and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$ and $\angle(A, C, B) < \pi$. Then $|F - E| = 4 \cdot \varnothing_\perp (A, B, C) \cdot \sin(\angle(A, C, B)/3) \cdot \sin(\angle(C, B, A)/3) \cdot \sin(\angle(B, A, C)/3).

**Proof:** $\sin((\pi/3) - (\angle(A, C, B)/3)) \neq 0$. $\sin((\pi/3) - (\angle(C, B, A)/3)) \neq 0$. $0 < \angle(A, C, B)$. $\angle(C, B, A) < \pi$. $0 < \angle(A, C, B) < \pi$ and $A$, $C$, $B$ are mutually different. $\angle(B, A, C) < \pi$. $0 < \angle(B, A, E) < \pi$. $\angle(A, E, B) < \pi$.

Suppose $A$, $B$, $C$ form a triangle and $\angle(E, B, A) = \angle(C, B, A)/3$ and $\angle(B, A, E) = \angle(B, A, C)/3$. Then $A$, $B$, $E$ form a triangle. The theorem is a consequence of (1) and (2).

Suppose $A$, $B$, $C$ form a triangle and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$. Then $A$, $F$, $C$ form a triangle. The theorem is a consequence of (1) and (2).

(19) Suppose $A$, $B$, $C$ form a triangle and $\angle(A, C, B) < \pi$ and $\angle(E, B, A) = \angle(C, B, A)/3$ and $\angle(B, A, E) = \angle(B, A, C)/3$ and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$ and $\angle(C, B, G) = \angle(C, B, A)/3$ and $\angle(G, C, B) = \angle(A, C, B)/3$. Now we state the propositions:

(i) $|F - E| = 4 \cdot \varnothing_\perp (A, B, C) \cdot \sin(\angle(A, C, B)/3) \cdot \sin(\angle(C, B, A)/3) \cdot \sin(\angle(B, A, C)/3)$, and

(ii) $|G - F| = 4 \cdot \varnothing_\perp (C, A, B) \cdot \sin(\angle(C, B, A)/3) \cdot \sin(\angle(B, A, C)/3) \cdot \sin(\angle(A, C, B)/3)$, and

(iii) $|E - G| = 4 \cdot \varnothing_\perp (B, C, A) \cdot \sin(\angle(B, A, C)/3) \cdot \sin(\angle(A, C, B)/3) \cdot \sin(\angle(C, B, A)/3)$.

The theorem is a consequence of (17), (18), (19), (2), (5), and (16).

(21) (i) $|F - E| = |G - F|$, and

(ii) $|F - E| = |E - G|$, and

(iii) $|G - F| = |E - G|$.
The theorem is a consequence of (20).

(22) **Morley’s Trisector Theorem:**
Suppose $A$, $B$, $C$ form a triangle and $\angle(A, B, C) < \pi$ and $\angle(E, C, A) = \angle(B, C, A)/3$ and $\angle(C, A, E) = \angle(C, A, B)/3$ and $\angle(A, B, F) = \angle(A, B, C)/3$ and $\angle(A, B, C) = \angle(C, A, B)/3$ and $\angle(C, A, E) = \angle(C, A, B)/3$ and $\angle(B, C, G) = \angle(B, C, A)/3$ and $\angle(G, B, C) = \angle(B, C, A)/3$. Then

(i) $|F - E| = |G - F|$, and

(ii) $|F - E| = |E - G|$, and

(iii) $|G - F| = |E - G|$. The theorem is a consequence of (21).

**References**


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