Mazur-Ulam Theorem

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**Summary.** The Mazur-Ulam theorem [15] has been formulated as two registrations: cluster bijective isometric -> midpoints-preserving Function of E,F; and cluster isometric midpoints-preserving -> Affine Function of E,F; A proof given by Jussi Väisälä [23] has been formalized.

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The notation and terminology used in this paper have been introduced in the following papers: [19], [18], [4], [5], [20], [11], [10], [14], [17], [1], [6], [16], [24], [25], [21], [13], [12], [22], [2], [9], [8], [3], and [7].

For simplicity, we use the following convention: E, F, G are real normed spaces, f is a function from E into F, g is a function from F into G, a, b are points of E, and t is a real number.

Let us note that I is closed.

Next we state four propositions:

1. **DYADIC** is a dense subset of I.
2. **DYADIC** = [0, 1].
3. \(a + a = 2 \cdot a\).
4. \((a + b) - b = a\).

Let A be an upper bounded real-membered set and let \(r\) be a non negative real number. Observe that \(r \circ A\) is upper bounded.

Let A be an upper bounded real-membered set and let \(r\) be a non positive real number. Note that \(r \circ A\) is lower bounded.

Let A be a lower bounded real-membered set and let \(r\) be a non negative real number. Observe that \(r \circ A\) is lower bounded.
Let $A$ be a lower bounded non-empty real-membered set and let $r$ be a non-positive real number. One can check that $r \circ A$ is upper bounded.

Next we state three propositions:

(5) For every sequence $f$ of real numbers holds $f + (\mathbb{N} \mapsto t) = t + f$.

(6) For every real number $r$ holds $\lim(\mathbb{N} \mapsto r) = r$.

(7) For every convergent sequence $f$ of real numbers holds $\lim(t + f) = t + \lim f$.

Let $f$ be a convergent sequence of real numbers and let us consider $t$. One can check that $t + f$ is convergent.

Next we state three propositions:

(8) For every sequence $f$ of real numbers holds $f \cdot (\mathbb{N} \mapsto a) = f \cdot a$.

(9) $\lim(\mathbb{N} \mapsto a) = a$.

(10) For every convergent sequence $f$ of real numbers holds $\lim(f \cdot a) = \lim f \cdot a$.

Let $f$ be a convergent sequence of real numbers and let us consider $E, a$.

Note that $f \cdot a$ is convergent.

Let $E, F$ be non-empty normed structures and let $f$ be a function from $E$ into $F$. We say that $f$ is isometric if and only if:

(Def. 1) For all points $a, b$ of $E$ holds $\|f(a) - f(b)\| = \|a - b\|$.

Let $E, F$ be non-empty RLS structures and let $f$ be a function from $E$ into $F$. We say that $f$ is affine if and only if:

(Def. 2) For all points $a, b$ of $E$ and for every real number $t$ such that $0 \leq t \leq 1$ holds $f((1 - t) \cdot a + t \cdot b) = (1 - t) \cdot f(a) + t \cdot f(b)$.

We say that $f$ preserves midpoints if and only if:

(Def. 3) For all points $a, b$ of $E$ holds $f(\frac{1}{2} \cdot (a + b)) = \frac{1}{2} \cdot (f(a) + f(b))$.

Let $E$ be a non-empty normed structure. Observe that $\text{id}_E$ is isometric.

Let $E$ be a non-empty RLS structure. Note that $\text{id}_E$ is affine and preserves midpoints.

Let $E$ be a non-empty normed structure. Observe that there exists a unary operation on $E$ which is bijective, isometric, and affine and preserves midpoints.

Next we state the proposition

(11) If $f$ is isometric and $g$ is isometric, then $g \cdot f$ is isometric.

Let us consider $E$ and let $f, g$ be isometric unary operations on $E$. One can verify that $g \cdot f$ is isometric.

The following proposition is true

(12) If $f$ is bijective and isometric, then $f^{-1}$ is isometric.

Let us consider $E$ and let $f$ be a bijective isometric unary operation on $E$. One can check that $f^{-1}$ is isometric.

We now state the proposition
(13) If $f$ preserves midpoints and $g$ preserves midpoints, then $g \cdot f$ preserves midpoints.

Let us consider $E$ and let $f$, $g$ be unary operations on $E$ preserving midpoints. Note that $g \cdot f$ preserves midpoints.

The following proposition is true

(14) If $f$ is bijective and preserves midpoints, then $f^{-1}$ preserves midpoints.

Let us consider $E$ and let $f$ be a bijective unary operation on $E$ preserving midpoints. Observe that $f^{-1}$ preserves midpoints.

Next we state the proposition

(15) If $f$ is affine and $g$ is affine, then $g \cdot f$ is affine.

Let us consider $E$ and let $f$, $g$ be affine unary operations on $E$. Observe that $g \cdot f$ is affine.

One can prove the following proposition

(16) If $f$ is bijective and affine, then $f^{-1}$ is affine.

Let us consider $E$ and let $f$ be a bijective affine unary operation on $E$. Observe that $f^{-1}$ is affine.

Let $E$ be a non empty RLS structure and let $a$ be a point of $E$. The functor $a$-reflection yields a unary operation on $E$ and is defined as follows:

(Def. 4) For every point $b$ of $E$ holds $a$-reflection($b$) = $2 \cdot a - b$.

The following proposition is true

(17) $a$-reflection \cdot $a$-reflection = id$_E$.

Let us consider $E$, $a$. Note that $a$-reflection is bijective.

We now state several propositions:

(18) $a$-reflection($a$) = $a$ and for every $b$ such that $a$-reflection($b$) = $b$ holds $a = b$.

(19) $a$-reflection($b$) - $a$ = $a - b$.

(20) $\|a$-reflection($b$) - $a\| = \|b - a\|$.

(21) $a$-reflection($b$) - $b$ = $2 \cdot (a - b)$.

(22) $\|a$-reflection($b$) - $b\| = 2 \cdot \|b - a\|$.

(23) $a$-reflection$^{-1}$ = $a$-reflection.

Let us consider $E$, $a$. Observe that $a$-reflection is isometric.

Next we state the proposition

(24) If $f$ is isometric, then $f$ is continuous on dom $f$.

Let us consider $E$, $F$. Observe that every function from $E$ into $F$ which is bijective and isometric also preserves midpoints.

Let us consider $E$, $F$. One can check that every function from $E$ into $F$ which is isometric and preserves midpoints is also affine.
REFERENCES


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