

Heron's Formula and Ptolemy's Theorem

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Summary. The goal of this article is to formalize some theorems that are in the [17] on the web. These are elementary theorems included in every handbook of Euclidean geometry and trigonometry: the law of cosines, the Heron's formula, the isosceles triangle theorem, the intersecting chords theorem and the Ptolemy's theorem.

MML identifier: EUCLID-6, version: 7.8.09 4.97.1001

The terminology and notation used here are introduced in the following articles: [5], [16], [2], [1], [13], [14], [15], [18], [12], [6], [8], [7], [11], [4], [9], [10], and [3].

1. LAW OF COSINES AND MEISTER-GAUSS FORMULA

We adopt the following rules: $p_1, p_2, p_3, p_4, p_5, p_6, p, p_7$ denote points of \mathcal{E}_T^2 and a, b, c, r, s denote real numbers.

Next we state four propositions:

- (1) If $\sin \angle(p_1, p_2, p_3) = \sin \angle(p_4, p_5, p_6)$ and $\cos \angle(p_1, p_2, p_3) = \cos \angle(p_4, p_5, p_6)$, then $\angle(p_1, p_2, p_3) = \angle(p_4, p_5, p_6)$.
- (2) $\sin \angle(p_1, p_2, p_3) = -\sin \angle(p_3, p_2, p_1)$.
- (3) $\cos \angle(p_1, p_2, p_3) = \cos \angle(p_3, p_2, p_1)$.
- (4) $\angle(p_1, p_4, p_2) + \angle(p_2, p_4, p_3) = \angle(p_1, p_4, p_3)$ or $\angle(p_1, p_4, p_2) + \angle(p_2, p_4, p_3) = \angle(p_1, p_4, p_3) + 2 \cdot \pi$.

Let us consider p_1, p_2, p_3 . The area of $\Delta(p_1, p_2, p_3)$ yields a real number and is defined by:

(Def. 1) The area of $\Delta(p_1, p_2, p_3) = \frac{1}{2} \cdot (((p_1)_1 \cdot (p_2)_2 - (p_2)_1 \cdot (p_1)_2) + ((p_2)_1 \cdot (p_3)_2 - (p_3)_1 \cdot (p_2)_2) + ((p_3)_1 \cdot (p_1)_2 - (p_1)_1 \cdot (p_3)_2))$.

Let us consider p_1, p_2, p_3 . The perimeter of $\Delta(p_1, p_2, p_3)$ yields a real number and is defined by:

(Def. 2) The perimeter of $\Delta(p_1, p_2, p_3) = |p_2 - p_1| + |p_3 - p_2| + |p_1 - p_3|$.

One can prove the following three propositions:

- (5) The area of $\Delta(p_1, p_2, p_3) = \frac{|p_1 - p_2| \cdot |p_3 - p_2| \cdot \sin \angle(p_3, p_2, p_1)}{2}$.
- (6) If $p_2 \neq p_1$, then $|p_3 - p_2| \cdot \sin \angle(p_3, p_2, p_1) = |p_3 - p_1| \cdot \sin \angle(p_2, p_1, p_3)$.
- (7) $(|p_3 - p_1|)^2 = ((|p_1 - p_2|)^2 + (|p_3 - p_2|)^2) - 2 \cdot (|p_1 - p_2|) \cdot (|p_3 - p_2|) \cdot \cos \angle(p_1, p_2, p_3)$.

2. SOME ELEMENTARY FACTS ABOUT EUCLIDEAN GEOMETRY

Next we state a number of propositions:

- (8) If $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_1$ and $p \neq p_2$, then $\angle(p_1, p, p_2) = \pi$.
- (9) If $p \in \mathcal{L}(p_2, p_3)$ and $p \neq p_2$, then $\angle(p_3, p_2, p_1) = \angle(p, p_2, p_1)$.
- (10) If $p \in \mathcal{L}(p_2, p_3)$ and $p \neq p_2$, then $\angle(p_1, p_2, p_3) = \angle(p_1, p_2, p)$.
- (11) If $\angle(p_1, p, p_2) = \pi$, then $p \in \mathcal{L}(p_1, p_2)$.
- (12) If $p \in \mathcal{L}(p_1, p_3)$ and $p \in \mathcal{L}(p_1, p_4)$ and $p_3 \neq p_4$ and $p \neq p_1$, then $p_3 \in \mathcal{L}(p_1, p_4)$ or $p_4 \in \mathcal{L}(p_1, p_3)$.
- (13) If $p \in \mathcal{L}(p_1, p_3)$ and $p \neq p_1$ and $p \neq p_3$, then $\angle(p_1, p, p_2) + \angle(p_2, p, p_3) = \pi$ or $\angle(p_1, p, p_2) + \angle(p_2, p, p_3) = 3 \cdot \pi$.
- (14) If $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_1$ and $p \neq p_2$ and $\angle(p_3, p, p_1) = \frac{\pi}{2}$ or $\angle(p_3, p, p_1) = \frac{3}{2} \cdot \pi$, then $\angle(p_1, p, p_3) = \angle(p_3, p, p_2)$.
- (15) If $p \in \mathcal{L}(p_1, p_3)$ and $p \in \mathcal{L}(p_2, p_4)$ and $p \neq p_1$ and $p \neq p_2$ and $p \neq p_3$ and $p \neq p_4$, then $\angle(p_1, p, p_2) = \angle(p_3, p, p_4)$.
- (16) If $|p_3 - p_1| = |p_2 - p_3|$ and $p_1 \neq p_2$, then $\angle(p_3, p_1, p_2) = \angle(p_1, p_2, p_3)$.
- (17) For all p_1, p_2, p_3, p such that $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_2$ holds $|(p_3 - p, p_2 - p_1)| = 0$ iff $|(p_3 - p, p_2 - p)| = 0$.
- (18) If $|p_1 - p_3| = |p_2 - p_3|$ and $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_3$ and $p \neq p_1$ and $\angle(p_3, p, p_1) = \frac{\pi}{2}$ or $\angle(p_3, p, p_1) = \frac{3}{2} \cdot \pi$, then $\angle(p_1, p_3, p) = \angle(p, p_3, p_2)$.
- (19) Let given p_1, p_2, p_3, p such that $|p_1 - p_3| = |p_2 - p_3|$ and $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_3$. Then
 - (i) if $\angle(p_1, p_3, p) = \angle(p, p_3, p_2)$, then $|p_1 - p| = |p - p_2|$,
 - (ii) if $|p_1 - p| = |p - p_2|$, then $|(p_3 - p, p_2 - p_1)| = 0$, and
 - (iii) if $|(p_3 - p, p_2 - p_1)| = 0$, then $\angle(p_1, p_3, p) = \angle(p, p_3, p_2)$.

Let us consider p_1, p_2, p_3 . We say that p_1, p_2 and p_3 are collinear if and only if:

(Def. 3) $p_1 \in \mathcal{L}(p_2, p_3)$ or $p_2 \in \mathcal{L}(p_3, p_1)$ or $p_3 \in \mathcal{L}(p_1, p_2)$.

Let us consider p_1, p_2, p_3 . We introduce p_1, p_2, p_3 form a triangle as an antonym of p_1, p_2 and p_3 are collinear.

The following propositions are true:

- (20) p_1, p_2, p_3 form a triangle iff p_1, p_2, p_3 are mutually different and $\angle(p_1, p_2, p_3) \neq \pi$ and $\angle(p_2, p_3, p_1) \neq \pi$ and $\angle(p_3, p_1, p_2) \neq \pi$.
- (21) Suppose p_1, p_2, p_3 form a triangle and p_4, p_5, p_6 form a triangle and $\angle(p_1, p_2, p_3) = \angle(p_4, p_5, p_6)$ and $\angle(p_3, p_1, p_2) = \angle(p_6, p_4, p_5)$. Then $|p_3 - p_2| \cdot |p_4 - p_6| = |p_1 - p_3| \cdot |p_6 - p_5|$ and $|p_3 - p_2| \cdot |p_5 - p_4| = |p_2 - p_1| \cdot |p_6 - p_5|$ and $|p_1 - p_3| \cdot |p_5 - p_4| = |p_2 - p_1| \cdot |p_4 - p_6|$.
- (22) Suppose p_1, p_2, p_3 form a triangle and p_4, p_5, p_6 form a triangle and $\angle(p_1, p_2, p_3) = \angle(p_4, p_5, p_6)$ and $\angle(p_3, p_1, p_2) = \angle(p_5, p_6, p_4)$. Then $|p_2 - p_3| \cdot |p_4 - p_6| = |p_3 - p_1| \cdot |p_5 - p_4|$ and $|p_2 - p_3| \cdot |p_6 - p_5| = |p_1 - p_2| \cdot |p_5 - p_4|$ and $|p_3 - p_1| \cdot |p_6 - p_5| = |p_1 - p_2| \cdot |p_4 - p_6|$.
- (23) If p_1, p_2, p_3 are mutually different and $\angle(p_1, p_2, p_3) \leq \pi$, then $\angle(p_2, p_3, p_1) \leq \pi$ and $\angle(p_3, p_1, p_2) \leq \pi$.
- (24) If p_1, p_2, p_3 are mutually different and $\angle(p_1, p_2, p_3) > \pi$, then $\angle(p_2, p_3, p_1) > \pi$ and $\angle(p_3, p_1, p_2) > \pi$.
- (25) If $p \in \mathcal{L}(p_1, p_2)$ and p_1, p_2, p_3 form a triangle and $\angle(p_1, p_3, p_2) = \angle(p, p_3, p_2)$, then $p = p_1$.
- (26) If $p \in \mathcal{L}(p_1, p_2)$ and $p_3 \notin \mathcal{L}(p_1, p_2)$ and $\angle(p_1, p_3, p_2) \leq \pi$, then $\angle(p, p_3, p_2) \leq \angle(p_1, p_3, p_2)$.
- (27) If $p \in \mathcal{L}(p_1, p_2)$ and $p_3 \notin \mathcal{L}(p_1, p_2)$ and $\angle(p_1, p_3, p_2) > \pi$ and $p \neq p_2$, then $\angle(p, p_3, p_2) \geq \angle(p_1, p_3, p_2)$.
- (28) If $p \in \mathcal{L}(p_1, p_2)$ and $p_3 \notin \mathcal{L}(p_1, p_2)$, then there exists p_4 such that $p_4 \in \mathcal{L}(p_1, p_2)$ and $\angle(p_1, p_3, p_4) = \angle(p, p_3, p_2)$.
- (29) If $p_1 \in \text{InsideOfCircle}(a, b, r)$ and $p_2 \in \text{OutsideOfCircle}(a, b, r)$, then there exists p such that $p \in \mathcal{L}(p_1, p_2) \cap \text{Circle}(a, b, r)$.
- (30) If $p_1, p_3, p_4 \in \text{Circle}(a, b, r)$ and $p \in \mathcal{L}(p_1, p_3)$ and $p \in \mathcal{L}(p_1, p_4)$ and $p_3 \neq p_4$, then $p = p_1$.
- (31) If $p_1, p_2, p \in \text{Circle}(a, b, r)$ and $p_7 = [a, b]$ and $p_7 \in \mathcal{L}(p, p_2)$ and $p_1 \neq p$, then $2 \cdot \angle(p_1, p, p_2) = \angle(p_1, p_7, p_2)$ or $2 \cdot (\angle(p_1, p, p_2) - \pi) = \angle(p_1, p_7, p_2)$.
- (32) If $p_1 \in \text{Circle}(a, b, r)$ and $r > 0$, then there exists p_2 such that $p_1 \neq p_2$ and $p_2 \in \text{Circle}(a, b, r)$ and $[a, b] \in \mathcal{L}(p_1, p_2)$.
- (33) If $p_1, p_2, p \in \text{Circle}(a, b, r)$ and $p_7 = [a, b]$ and $p_1 \neq p$ and $p_2 \neq p$, then $2 \cdot \angle(p_1, p, p_2) = \angle(p_1, p_7, p_2)$ or $2 \cdot (\angle(p_1, p, p_2) - \pi) = \angle(p_1, p_7, p_2)$.
- (34) Suppose $p_1, p_2, p_3, p_4 \in \text{Circle}(a, b, r)$ and $p_1 \neq p_3$ and $p_1 \neq p_4$ and $p_2 \neq p_3$ and $p_2 \neq p_4$. Then $\angle(p_1, p_3, p_2) = \angle(p_1, p_4, p_2)$ or $\angle(p_1, p_3, p_2) = \angle(p_1, p_4, p_2) - \pi$ or $\angle(p_1, p_3, p_2) = \angle(p_1, p_4, p_2) + \pi$.
- (35) If $p_1, p_2, p_3 \in \text{Circle}(a, b, r)$ and $p_1 \neq p_2 \neq p_3$, then $\angle(p_1, p_2, p_3) \neq \pi$.

- (36) Suppose $p_1, p_2, p_3, p_4 \in \text{Circle}(a, b, r)$ and $p \in \mathcal{L}(p_1, p_3)$ and $p \in \mathcal{L}(p_2, p_4)$ and p_1, p_2, p_3, p_4 are mutually different. Then $\sphericalangle(p_1, p_4, p_2) = \sphericalangle(p_1, p_3, p_2)$.
- (37) If $p_1, p_2, p_3 \in \text{Circle}(a, b, r)$ and $\sphericalangle(p_1, p_2, p_3) = 0$ and $p_1 \neq p_2 \neq p_3$, then $p_1 = p_3$.
- (38) If $p_1, p_2, p_3, p_4 \in \text{Circle}(a, b, r)$ and $p \in \mathcal{L}(p_1, p_3)$ and $p \in \mathcal{L}(p_2, p_4)$, then $|p_1 - p| \cdot |p - p_3| = |p_2 - p| \cdot |p - p_4|$.

3. HERON'S FORMULA AND PTOLEMY'S THEOREM

One can prove the following propositions:

- (39) Suppose $a = |p_2 - p_1|$ and $b = |p_3 - p_2|$ and $c = |p_1 - p_3|$ and $s = \frac{1}{2} \cdot$ the perimeter of $\Delta(p_1, p_2, p_3)$. Then $|\text{the area of } \Delta(p_1, p_2, p_3)| = \sqrt{s \cdot (s - a) \cdot (s - b) \cdot (s - c)}$.
- (40) If $p_1, p_2, p_3, p_4 \in \text{Circle}(a, b, r)$ and $p \in \mathcal{L}(p_1, p_3)$ and $p \in \mathcal{L}(p_2, p_4)$, then $|p_3 - p_1| \cdot |p_4 - p_2| = |p_2 - p_1| \cdot |p_4 - p_3| + |p_3 - p_2| \cdot |p_4 - p_1|$.

4. APPENDIX

In the sequel c_1, c_2, c_3 denote elements of \mathbb{C} .

One can prove the following propositions:

- (41) $(p_1 - p_2)_1 = (p_1)_1 - (p_2)_1$ and $(p_1 - p_2)_2 = (p_1)_2 - (p_2)_2$.
- (42) $|p_1 - p_2| = 0$ iff $p_1 = p_2$.
- (43) $|p_1 - p_2| = |p_2 - p_1|$.
- (44) $\sphericalangle(p_1, p_2, p_3) \neq 2 \cdot \sphericalangle(p_4, p_5, p_6) + 2 \cdot \pi$.
- (45) $\sphericalangle(p_1, p_2, p_3) \neq 2 \cdot \sphericalangle(p_4, p_5, p_6) + 4 \cdot \pi$.
- (46) $\sphericalangle(p_1, p_2, p_3) \neq 2 \cdot \sphericalangle(p_4, p_5, p_6) - 4 \cdot \pi$.
- (47) $\sphericalangle(p_1, p_2, p_3) \neq 2 \cdot \sphericalangle(p_4, p_5, p_6) - 6 \cdot \pi$.
- (48) $\sphericalangle(p_1, p_2, p_3) = \sphericalangle((\text{euc2cpx}(p_1 - p_2)), (\text{euc2cpx}(p_3 - p_2)))$.
- (49) $\sphericalangle(c_1, c_2) + \sphericalangle(c_2, c_3) = \sphericalangle(c_1, c_3)$ or $\sphericalangle(c_1, c_2) + \sphericalangle(c_2, c_3) = \sphericalangle(c_1, c_3) + 2 \cdot \pi$.
- (50) Suppose $c_1 = \text{euc2cpx}(p_1 - p_2)$ and $c_2 = \text{euc2cpx}(p_3 - p_2)$. Then $\Re((c_1|c_2)) = ((p_1)_1 - (p_2)_1) \cdot ((p_3)_1 - (p_2)_1) + ((p_1)_2 - (p_2)_2) \cdot ((p_3)_2 - (p_2)_2)$ and $\Im((c_1|c_2)) = -((p_1)_1 - (p_2)_1) \cdot ((p_3)_2 - (p_2)_2) + ((p_1)_2 - (p_2)_2) \cdot ((p_3)_1 - (p_2)_1)$ and $|c_1| = \sqrt{((p_1)_1 - (p_2)_1)^2 + ((p_1)_2 - (p_2)_2)^2}$ and $|p_1 - p_2| = |c_1|$.
- (51) Let n be an element of \mathbb{N} , q_1 be a point of \mathcal{E}_T^n , and f be a function from \mathcal{E}_T^n into \mathbb{R}^1 . If for every point q of \mathcal{E}_T^n holds $f(q) = |q - q_1|$, then f is continuous.

- (52) Let n be an element of \mathbb{N} and q_1 be a point of \mathcal{E}_T^n . Then there exists a function f from \mathcal{E}_T^n into \mathbb{R}^1 such that for every point q of \mathcal{E}_T^n holds $f(q) = |q - q_1|$ and f is continuous.

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Received January 10, 2008
