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Linearity of Lebesgue Integral of Simple Valued Function[1]

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Summary. In this article the authors prove linearity of the Lebesgue integral of simple valued function.

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The notation and terminology used here are introduced in the following papers: [16], [17], [1], [15], [2], [18], [7], [9], [8], [3], [4], [5], [6], [10], [11], [12], [14], and [13].

One can prove the following propositions:

(1) Let F, G, H be finite sequences of elements of $\overline{\mathbb{R}}$. Suppose that

(i) for every natural number i such that $i \in \operatorname{dom} F$ holds $0_{\overline{\mathbb{R}}} \leq F(i)$,

(ii) for every natural number i such that $i \in \operatorname{dom} G$ holds $0_{\overline{\mathbb{R}}} \leq G(i)$,

(iii) $\operatorname{dom} F = \operatorname{dom} G$, and

(iv) $H = F + G$.

Then $\sum H = \sum F + \sum G$.

(2) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, n be a natural number, f be a partial function from X to $\overline{\mathbb{R}}$, F be a finite sequence of separated subsets of S, and a, x be finite sequences of elements of $\overline{\mathbb{R}}$. Suppose that f is simple function in S and $\operatorname{dom} f \neq \emptyset$ and for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$ and F and a are representation of f and $\operatorname{dom} x = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} x$ holds $x(i) = a(i) \cdot (M \cdot F)(i)$ and $\operatorname{len} F = n$. Then $\int\limits_X f \, \mathrm{d}M = \sum x$.

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(3) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, M be a σ-measure on S, F be a finite sequence of separated subsets of S, and a, x be finite sequences of elements of $\overline{\mathbb{R}}$. Suppose that

(i) f is simple function in S,

(ii) $\operatorname{dom} f \neq \emptyset$,

(iii) for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$,

(iv) F and a are representation of f,

(v) $\operatorname{dom} x = \operatorname{dom} F$, and

(vi) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$.

Then $\int\limits_X f \, \mathrm{d}M = \sum x$.

(4) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, and M be a σ-measure on S. Suppose f is simple function in S and $\operatorname{dom} f \neq \emptyset$ and for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$. Then there exists a finite sequence F of separated subsets of S and there exist finite sequences a, x of elements of $\overline{\mathbb{R}}$ such that

(i) F and a are representation of f,

(ii) $\operatorname{dom} x = \operatorname{dom} F$,

(iii) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$, and

(iv) $\int\limits_X f \, \mathrm{d}M = \sum x$.

(5) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) f is simple function in S,

(ii) $\operatorname{dom} f \neq \emptyset$,

(iii) for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$,

(iv) g is simple function in S,

(v) $\operatorname{dom} g = \operatorname{dom} f$, and

(vi) for every set x such that $x \in \operatorname{dom} g$ holds $0_{\overline{\mathbb{R}}} \leq g(x)$.

Then

(vii) $f + g$ is simple function in S,

(viii) $\operatorname{dom}(f + g) \neq \emptyset$,

(ix) for every set x such that $x \in \operatorname{dom}(f + g)$ holds $0_{\overline{\mathbb{R}}} \leq (f + g)(x)$, and

(x) $\int\limits_X f + g \, \mathrm{d}M = \int\limits_X f \, \mathrm{d}M + \int\limits_X g \, \mathrm{d}M$.

(6) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f, g be partial functions from X to $\overline{\mathbb{R}}$, and c be an extended real number. Suppose that f is simple function in S and $\operatorname{dom} f \neq \emptyset$ and for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$ and $0_{\overline{\mathbb{R}}} \leq c$ and

$c < +\infty$ and $\operatorname{dom} g = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom} g$ holds $g(x) = c \cdot f(x)$. Then $\int_X g \, \mathrm{d}M = c \cdot \int_X f \, \mathrm{d}M$.

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