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Yoneda Embedding

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The articles [7], [10], [8], [11], [1], [2], [3], [4], [6], [5], and [9] provide the notation and terminology for this paper.

In this paper A is a category, a is an object of A, and f is a morphism of A.

Let us consider A. The functor $\mathrm{EnsHom}A$ yields a category and is defined by:

(Def. 1) $\mathrm{EnsHom}A = \mathbf{Ens}_{\mathrm{Hom}(A)}$.

The following propositions are true:

(1) Let f, g be functions and m_1, m_2 be morphisms of $\mathrm{EnsHom}A$. If $\mathrm{cod}\, m_1 = \mathrm{dom}\, m_2$ and $\langle\langle \mathrm{dom}\, m_1, \mathrm{cod}\, m_1\rangle, f\rangle = m_1$ and $\langle\langle \mathrm{dom}\, m_2, \mathrm{cod}\, m_2\rangle, g\rangle = m_2$, then $\langle\langle \mathrm{dom}\, m_1, \mathrm{cod}\, m_2\rangle, g \cdot f\rangle = m_2 \cdot m_1$.

(2) $\mathrm{hom}(a, -)$ is a functor from A to $\mathrm{EnsHom}A$.

Let us consider A, a. The functor $\mathrm{hom}^{\mathrm{F}}(a, -)$ yields a functor from A to $\mathrm{EnsHom}A$ and is defined as follows:

(Def. 2) $\mathrm{hom}^{\mathrm{F}}(a, -) = \mathrm{hom}(a, -)$.

The following proposition is true

(3) For every morphism f of A holds $\mathrm{hom}^{\mathrm{F}}(\mathrm{cod}\, f, -)$ is naturally transformable to $\mathrm{hom}^{\mathrm{F}}(\mathrm{dom}\, f, -)$.

Let us consider A, f. The functor $\mathrm{hom}^{\mathrm{F}}(f, -)$ yielding a natural transformation from $\mathrm{hom}^{\mathrm{F}}(\mathrm{cod}\, f, -)$ to $\mathrm{hom}^{\mathrm{F}}(\mathrm{dom}\, f, -)$ is defined as follows:

(Def. 3) For every object o of A holds $(\mathrm{hom}^{\mathrm{F}}(f, -))(o) = \langle\langle \mathrm{hom}(\mathrm{cod}\, f, o), \mathrm{hom}(\mathrm{dom}\, f, o)\rangle, \mathrm{hom}(f, \mathrm{id}_o)\rangle$.

We now state the proposition

(4) For every element f of the morphisms of A holds $\langle\langle \mathrm{hom}^{\mathrm{F}}(\mathrm{cod}\, f, -), \mathrm{hom}^{\mathrm{F}}(\mathrm{dom}\, f, -)\rangle, \mathrm{hom}^{\mathrm{F}}(f, -)\rangle$ is an element of the morphisms of $(\mathrm{EnsHom}A)^A$.

Let us consider A. The functor $\mathrm{Yoneda}A$ yielding a contravariant functor from A into $(\mathrm{EnsHom}A)^A$ is defined as follows:

(Def. 4) For every morphism f of A holds $(\mathrm{Yoneda}A)(f) = \langle\langle \mathrm{hom}^{\mathrm{F}}(\mathrm{cod}\, f, -), \mathrm{hom}^{\mathrm{F}}(\mathrm{dom}\, f, -)\rangle, \mathrm{hom}^{\mathrm{F}}(f, -)\rangle$.

Let A, B be categories, let F be a contravariant functor from A into B, and let c be an object of A. The functor $F(c)$ yields an object of B and is defined by:

(Def. 5) $F(c) = (\mathrm{Obj}\,F)(c)$.

One can prove the following proposition

(5) For every functor F from A to $(\mathrm{EnsHom}A)^A$ such that $\mathrm{Obj}\,F$ is one-to-one and F is faithful holds F is one-to-one.

Let C, D be categories and let T be a contravariant functor from C into D. We say that T is faithful if and only if:

(Def. 6) For all objects c, c' of C such that $\mathrm{hom}(c,c') \neq \emptyset$ and for all morphisms f_1, f_2 from c to c' such that $T(f_1) = T(f_2)$ holds $f_1 = f_2$.

The following propositions are true:

(6) Let F be a contravariant functor from A into $(\mathrm{EnsHom}A)^A$. If $\mathrm{Obj}\,F$ is one-to-one and F is faithful, then F is one-to-one.

(7) Yoneda$A$ is faithful.

(8) Yoneda$A$ is one-to-one.

Let C, D be categories and let T be a contravariant functor from C into D. We say that T is full if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let c, c' be objects of C. Suppose $\mathrm{hom}(T(c'),T(c)) \neq \emptyset$. Let g be a morphism from $T(c')$ to $T(c)$. Then $\mathrm{hom}(c,c') \neq \emptyset$ and there exists a morphism f from c to c' such that $g = T(f)$.

Next we state the proposition

(9) Yoneda$A$ is full.

REFERENCES

[1] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[3] Czesław Byliński. Introduction to categories and functors. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/cat_1.html`.

[4] Czesław Byliński. Subcategories and products of categories. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/cat_2.html`.

[5] Czesław Byliński. Category Ens. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/ens_1.html`.

[6] Czesław Byliński. Opposite categories and contravariant functors. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/oppcat_1.html`.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[8] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/mcart_1.html`.

[9] Andrzej Trybulec. Natural transformations. Discrete categories. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/nattra_1.html`.

[10] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[11] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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